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CORRESPONDENCE OF HERMITIAN MODULAR FORMS TO CYCLES ASSOCIATED TO $SU(p,2)$

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In the previous papers [11], [12] we have given a correspondence, in the form of a lifting through a theta function, from Hermitian modular forms of degree r to codimension r geodesic cycles in the locally symmetric spaces associated to $SU(p,1)$. Our purpose here is to extend this correspondence to the case where the target is associated to a unitary group of rank greater than one: $SU(p,2)$. In [11, §1] we have defined geodesic cycles of codimension $2r$, $1 \leqslant r \leqslant p-1$, associated to $SU(p,2)$. In this paper we make the further restriction that $r=1$. This has the merit that while exhibiting some of the new phenomena in higher rank, it is basically differential forms of degree (2, 2) we are dealing with and some steps are manageable by direct calculations.

The theory of Weil representation and theta functions for dual reductive pairs has been utilized to construct liftings of automorphic forms in abundant cases (cf. the references to [5], [8], [11], [12]). All these give correspondence of automorphic forms associated to two different groups. Other than some accidental lower dimensional cases it is considerably more surprising, and technically more subtle, that this machinery embodies a lift from automorphic forms to harmonic forms dual to special cycles. The technical subtleties appear inevitable since one is trying to link automorphic objects with higher dimensional geometric objects.

As first found for $SO(p,1)$ in [8], and then a modified version found for $SU(p,1)$ in [11], [12], this link to geometry comes from two ingredients.

(i) *A construction of the harmonic form dual to such a cycle as a special value via analytic continuation of a one (complex) parameter family of dual forms.* Let $\mathfrak{D}$ be the bounded symmetric domain for $G = SU(p,2)$ and Γ a cocompact discrete subgroup of G. In §1 we consider a particular subdomain $\mathfrak{D}_1 \subset \mathfrak{D}$ of codimension 2. Denote by $G_1 \subset G$ the subgroup leaving $\mathfrak{D}_1$ invariant and

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$\Gamma_1 = \Gamma \cap G_1$. The cycle of interest is the image of $\Gamma_1\backslash\mathfrak{D}_1$ via the projection $\Gamma_1\backslash\mathfrak{D} \to \Gamma\backslash\mathfrak{D}$. One constructs a family $\omega(s)$ of dual forms in $\Gamma_1\backslash\mathfrak{D}$ and then descends to the dual form in $\Gamma\backslash\mathfrak{D}$ by the sum

$$\hat{\omega}(s) = \sum_{\gamma\in\Gamma_1\backslash\Gamma} \gamma^*\omega(s).$$

The construction of $\omega(s)$, as in [11] (cf. [7] for a different method), is to compare the Bott-Chern singular forms of two canonical metrics on a natural vector bundle on $\Gamma_1\backslash\mathfrak{D}$, which has a section vanishing precisely on $\Gamma_1\backslash\mathfrak{D}_1$. The resulting explicit formula of $\omega(s)$ is in terms of G_1 invariant forms. For analytic continuation of $\hat{\omega}(s)$ we study the action of Laplacian on the various families of G_1 invariant forms. In general with increasing q (in $SU(p,q)$) and codimension the number of G_1 invariant forms increase rapidly and the calculations of their Laplacians become unwieldy (cf. Lemmas (1.8)–(1.11) and Proposition (1.12)).

(ii) *Identification of the special value* $\omega(p-1)$ *with polynomials which are admissible as the coefficients of certain theta functions.* For this the differential forms are pulled back from $\mathfrak{D}$ to a vector space and the main point in [12] as well as §2 below is a splitting of the appropriate tensors into $\mathcal{K} \oplus \mathfrak{L}$ (we have interchanged the symbols $\mathcal{K}$, $\mathfrak{L}$ of [12] so that $\mathcal{K}$ is in the kernel and $\mathfrak{L}$ is where lifting takes place) where (a) $\omega(p-1)$ vanishes on $\mathcal{K}$, and (b) the special values $\omega(p-1)|_{\mathfrak{L}}$ give precisely the admissible polynomials. It is also the differential operators corresponding to tensors in $\mathfrak{L}$ which have the correct action on the basic Schwartz function (cf. Proposition (2.10)). We carry through (a) and (b) by an explicit evaluation which is manageable in the present case. In general a systematic use of invariant theory seems necessary.

After (i) and (ii) it remains to compare the lifted family

$$\sum_{\gamma\in\Gamma_1\backslash\Gamma} \gamma^*\big((A/B)^{s-p+1}(\omega(p-1))\big)$$

(cf. (4.1)) with $\hat{\omega}(s)$. In the rank one case [12] there is a simple relation $\mathbf{P}(\omega(s)) = (A/B)^{s-p+1}\omega(p-1)$ where $\mathbf{P}$ is the projection to leading primitive component and is expressible in terms of the Kähler operators L, Λ. In the present higher rank case a new G invariant form α appears (other than the Kähler form and its powers cf. (4.14)(i)) and is essential to the construction of $\omega(s)$. It leads us to the following main result:

$$(*) \qquad \text{analytic continuation of} \left\{ \sum_{\gamma\in\Gamma_1\backslash\Gamma} \gamma^*\left(\left(\frac{A}{B}\right)^{s-p+1}\omega(p-1)\right)\right\}$$

$$\text{at } p-1 = P_\alpha(\hat{\omega}(p-1))$$

where P_α is the cohomology operation (4.14)(ii)

$$(**)\qquad P_\alpha(x) = x - \frac{1}{p+2}(\Lambda\alpha)(\Lambda x) + \frac{1}{p+2}\Lambda\{\alpha(\Lambda x)\} - \frac{1}{(p+1)(p+2)}\alpha(\Lambda^2 x).$$

This displays the lifting explicitly as a harmonic form. We also prove that $P_\alpha(\hat{\omega}(p-1))$ is *primitive* and is *orthogonal* in Hodge inner product *to all forms on* $\Gamma\backslash\mathfrak{D}$ *arising from* G *invariant forms*. The last result is a consequence of $\mathfrak{K}$ containing all the G invariant tensors.

From the present result it is reasonable to guess in general there is a lifting from Hermitian modular forms of degree r to the cohomology spanned by geodesic cycles of codimension rq associated to $SU(p, q)$. Furthermore the image should lie in the subspace which is both *primitive* and *orthogonal to all* G *invariant forms*. It is harder to see how the various G invariant forms intervene to bring a generalization of $(**)$.

The lifting derived from $(*)$ has as its main consequence (cf. (4.18)) the equality of Fourier coefficients of Hermitian cusp forms (in the image of adjoint of lifting) with intersection numbers (cf. [3], [6], [8]). Siegel has proved via his Main Theorem that volumes of analogous cycles in arithmetic quotients associated to indefinite quadratic forms are the Fourier coefficients of Eisenstein series (cf. his papers on indefinite quadratic forms, particularly [9, §§1, 12]). Siegel's results are exactly complementary to ours since volumes are obtained by pairing with powers of Kähler forms which vanishes in our case since we have primitive cohomology. This is also reflected in the cusp forms that we obtain in contrast to Siegel's Eisenstein series. We know of a more direct approach to Siegel's type of results based partly on the geometric forms developed here.

Finally it is intriguing to observe that the two ingredients (i) and (ii) described here are exactly analogous to the two steps involved in the proof of the Riemann Roch Theorem based on intersection theory [10]. Namely, there one makes a canonical construction of the dual cochain of a cycle and then uses invariant theory to identify the restrictions of this cochain. In fact, the general problem in (ii) is also to make fuller use of invariant theory. This analogy renders more interesting the question of the image and kernel of the lifting and perhaps also of the eventual applications of these correspondences. In this respect we note that the analogous Hirzebruch-Zagier cycles on Hilbert modular surfaces play an important role in the recent work of Harder, Langlands, and Rapoport on Tate's conjectures for these surfaces.

1. Harmonic forms dual to geodesic cycles

Let $G = SU(p, q)$ and $\mathfrak{D}$ be the symmetric space associated to G which is realized as the bounded domain

$$\mathfrak{D} = \left\{ Z \in M_{pq}(\mathbf{C}) \,\middle|\, {}^t Z \bar{Z} < E_q \right\}.$$

For $g \in G$, $g = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ with $A \in M_{pp}(\mathbf{C})$, $B \in M_{pq}(\mathbf{C})$, $C \in M_{qp}(\mathbf{C})$ and $D \in M_{qq}(\mathbf{C})$. G acts on $\mathfrak{D}$ by

$$gZ = (AZ + B)(CZ + D)^{-1}.$$

We have an automorphic factor $j(g, Z) = CZ + D$ and the linear action of G is related to its action on $\mathfrak{D}$ by

$$(1.1) \qquad g\begin{pmatrix} Z \\ E_q \end{pmatrix} = \begin{pmatrix} gZ \\ E_q \end{pmatrix} j(g, Z).$$

On $\mathfrak{D}$ the Kähler metric is

$$(1.2) \qquad \kappa = \operatorname{tr}\left((E - Z\,{}^t\bar{Z})^{-1} dZ (E - {}^t\bar{Z} Z)^{-1}\, {}^t d\bar{Z} \right).$$

We decompose Z as

$$(1.3) \qquad Z = \begin{pmatrix} Z_1 \\ Z_2 \end{pmatrix}$$

with $Z_1 \in M_{p-1,q}(\mathbf{C})$, $Z_2 \in M_{1q}(\mathbf{C})$ and denote by $\mathfrak{D}_1$ the subsymmetric domain

$$(1.4) \qquad \mathfrak{D}_1 = \{ Z \in \mathfrak{D} \mid Z_2 = 0 \}.$$

Let $e_1, \cdots, e_{p+q}$ be the standard basis of $\mathbf{C}^{p+q}$ and G_1 the isotropy subgroup of G at the line $\mathbf{C}e_p$. $g \in G_1$ has a block form

$$(1.5) \qquad g = \begin{pmatrix} A_1 & 0 & B_1 \\ 0 & \lambda & 0 \\ C_1 & 0 & D_1 \end{pmatrix}$$

with $A_1 \in M_{p-1\,p-1}(\mathbf{C})$, $D_1 \in M_{qq}(\mathbf{C})$, $|\lambda| = 1$. From [11, Lemma (1.2)] G_1 leaves $\mathfrak{D}_1$ invariant and its action on $\mathfrak{D}$ is given by

$$(1.6) \qquad gZ = \begin{pmatrix} (A_1 Z_1 + B_1)(C_1 Z_1 + D_1)^{-1} \\ \lambda Z_2 j^{-1}(g, Z) \end{pmatrix}, \qquad g \in G_1, Z \in \mathfrak{D}.$$

We order the coordinates of Z by $z_{11},\cdots,z_{1q}, z_{21},\cdots,z_{2q},\cdots,z_{p1},\cdots,z_{pq}$. At $Z_1 = 0$ the coefficient matrix of the metric (1.2) is given by

$$(1.7)\qquad (g_{\alpha\bar\beta}) = \begin{pmatrix} E_q + \dfrac{{}^tZ_2\bar Z_2}{1-|Z_2|^2} & \cdots & 0 & 0 \\ & \ddots & \vdots & \vdots \\ 0 & \cdots & E_q + \dfrac{{}^tZ_2\bar Z_2}{1-|Z_2|^2} & 0 \\ 0 & \cdots & 0 & \dfrac{1}{1-|Z_2|^2}\left(E_q + \dfrac{{}^tZ_2\bar Z_2}{1-|Z_2|^2}\right) \end{pmatrix}$$

and its inverse is

$$(1.8)\qquad (g^{\alpha\bar\beta}) = \begin{pmatrix} E_q - {}^tZ_2\bar Z_2 & \cdots & 0 \\ & \ddots & \\ \vdots & E_q - {}^tZ_2\bar Z_2 & \vdots \\ 0 & \cdots & (1-|Z_2|^2)(E_q - {}^tZ_2\bar Z_2) \end{pmatrix}$$

As in [11, §1] we introduce functions

$$(1.9)\qquad \begin{aligned} A(Z) &= \det(E - {}^tZ\bar Z), \\ B(Z) &= \det(E - {}^tZ_1\bar Z_1), \\ C(Z) &= B(Z) - A(Z). \end{aligned}$$

Now since

$$\begin{aligned} E - {}^tZ\bar Z &= E - {}^tZ_1\bar Z_1 - {}^tZ_2\bar Z_2 \\ &= (E - {}^tZ_1\bar Z_1)^{1/2}\{E - (E - {}^tZ_1\bar Z_1)^{-1/2}\,{}^tZ_2\bar Z_2(E - {}^tZ_1\bar Z_1)^{-1/2}\} \\ &\quad \cdot (E - {}^tZ_1\bar Z_1)^{1/2} \end{aligned}$$

we have

$$(1.10)\qquad \frac{A}{B}(Z) = 1 - \bar Z_2(E - {}^tZ_1\bar Z_1)^{-1}\,{}^tZ_2.$$

Let $H_Z = (E - {}^tZ_1\bar Z_1)^{-1}$ and $\tilde H_Z = (E - {}^tZ\bar Z)^{-1}$.

Lemma (1.1) [11].

$$H_{gZ} = \overline{j(g,Z)}\,H_Z\,{}^tj(g,Z), \qquad g \in G_1,$$

$$\tilde H_{gZ} = \overline{j(g,Z)}\,\tilde H_Z\,{}^tj(g,Z), \qquad g \in G.$$

It follows that B/A is G_1 invariant. By [11, Proposition (1.7)] B/A has the geometric meaning

$$\frac{B}{A}(Z) = \cosh^2 d(Z, \mathfrak{D}_1) \tag{1.11}$$

where $d(Z, \mathfrak{D}_1)$ is the distance from Z to $\mathfrak{D}_1$. Now consider the bundle $E = \mathfrak{D} \times \mathbf{C}^q$ where $\mathbf{C}^q$ is viewed as the space of complex column q-vectors. We define an action of G_1 on E. For

$$g = \begin{pmatrix} A_1 & 0 & B_1 \\ 0 & \lambda & 0 \\ C_1 & 0 & D_1 \end{pmatrix} \in G_1, \qquad (Z, Y) \in E,$$

$$g(Z, Y) = \left(gZ, \lambda^t j^{-1}(g, Z)Y\right). \tag{1.12}$$

On E, H_Z and $\tilde{H}_Z$ define Hermitian fiber metrics which give admissible connections of type $(1, 0)$. We have a canonical holomorphic section $v\colon \mathfrak{D} \to E$

$$v(Z) = \left(Z, {}^tZ_2\right), \qquad (Z \in \mathfrak{D}). \tag{1.13}$$

Note that the fiber metrics defined by H_Z and $\tilde{H}_Z$ respectively, and the holomorphic section v are all G_1 invariant. From the constructions of $\mathfrak{D}_1$ and v, it is easy to observe

$$\mathfrak{D}_1 = \{Z \in \mathfrak{D} \mid v(Z) = 0\}.$$

Let $C(E)$ and $\tilde{C}(E)$ be the q-th Chern forms of the Hermitian structures given by H_Z and $\tilde{H}_Z$ respectively. By the construction in [11, §2] there exist forms ψ and $\tilde{\psi}$, defined outside $\mathfrak{D}_1$, of type $(q, q-1)$ such that

$$\bar{\partial}\psi = C(E) \quad \text{and} \quad \bar{\partial}\tilde{\psi} = \tilde{C}(E).$$

We recall briefly the construction. The connection and curvature matrices are given by

$$\begin{aligned} \omega &= d^tZ_1\left(E - \bar{Z}_1{}^tZ_1\right)^{-1}\bar{Z}_1, \\ \tilde{\omega} &= d^tZ(E - \bar{Z}^tZ)^{-1}\bar{Z}, \\ \Omega &= -d^tZ_1\left(E - \bar{Z}_1{}^tZ_1\right)^{-1}d\bar{Z}_1\left(E - {}^tZ_1\bar{Z}_1\right)^{-1}, \\ \tilde{\Omega} &= -d^tZ(E - \bar{Z}^tZ)^{-1}d\bar{Z}\left(E - {}^tZ\bar{Z}\right)^{-1}. \end{aligned} \tag{1.14}$$

Let us introduce

$$e = (e_1 \cdots e_q),$$
$$v = (e_1 \cdots e_q)^t Z_2 = e_1 z_{p1} + \cdots + e_q z_{pq},$$
$$\alpha = dv d\bar{v}, \quad K = -e\Omega H^{-1}\, {}^t\bar{e},$$
$$s_k = \bar{v} dv \alpha^{k-1} K^{q-k}, \tag{1.15}$$
$$c_q \psi_k \chi\bar{\chi} = \frac{s_k}{|v|^{2k}}, \quad \psi = \sum_{k=1}^{q} (-1)^k \binom{q}{k} \psi_k,$$

where $\chi = B^{1/2} e_1 \cdots e_q$ and $c_q = (-1)^{q^2/2-1} q!(2\pi)^q$. There are also the corresponding objects $\tilde{\alpha}$, $\tilde{K}$, $\tilde{s}_k$, $\tilde{\psi}_k$ and $\tilde{\psi}$ for the metric $\tilde{H}$ given by the same formulas as (1.15). At $Z_1 = 0$

$$\omega = 0, \quad \tilde{\omega} = \frac{({}^t dZ_2)\bar{Z}_2}{1 - |Z_2|^2}, \quad dv = e^t dZ_2, \quad \tilde{d}v = \frac{B}{A} dv.$$

Hence

$$\bar{v}\tilde{d}v = \frac{B}{A}\bar{v} dv, \quad \tilde{\alpha} = \left(\frac{B}{A}\right)^2 \alpha \tag{1.16}$$

and also

$$\Omega H^{-1} = -{}^t dZ_1 \wedge d\bar{Z}_1, \quad \tilde{\Omega}\tilde{H}^{-1} = \Omega H^{-1} - \left(\frac{B}{A}\right) {}^t dZ_2 d\bar{Z}_2.$$

Consequently

$$\tilde{K} = K + \left(\frac{B}{A}\right)\alpha. \tag{1.17}$$

From (1.15)–(1.17) the following lemma is straightforward.

Lemma (1.2). $\tilde{\psi} = \psi - \sum_{k=1}^{q} \binom{q}{k} (C/A)^k \psi_k$.

For the rest of this section we assume $q = 2$. Regrouping the above formula we have

$$\tilde{\psi} = \left[2\left(\frac{B}{A}\right) - \left(\frac{B}{A}\right)^2\right]\psi - 2\left(\frac{B}{A}\right)^2 \left(\frac{C}{B}\psi_1\right). \tag{1.18}$$

This leads us to the form depending on $s \in \mathbf{C}$.

$$\Phi(s) = \frac{(A/B)^{s+2}}{s+2}\tilde{C}(E) + \left(\frac{-2(A/B)^{s+2}}{s+1} + \frac{(A/B)^s}{s}\right) C(E) + \frac{2(A/B)^s}{s}\bar{\partial}\left(\frac{C}{B}\psi_1\right). \tag{1.19}$$

Proposition (1.3). *Let Γ_1 be a discrete subgroup of G_1 such that $\Gamma_1 \backslash \mathfrak{D}_1$ is compact, and η a Γ_1 invariant $\bar{\partial}$ closed $(2(p-1), 2(p-1))$ form on $\mathfrak{D}$ with bounded norm. Then for* $\mathrm{Re}(s) > p+1$,

$$\int_{\Gamma_1 \backslash \mathfrak{D}} \Phi(s) \wedge \eta = \frac{s}{s(s+1)(s+2)} \int_{\Gamma_1 \backslash \mathfrak{D}_1} \eta.$$

Proof. Let

$$\psi(s) = \frac{(A/B)^{s+2}}{s+2}\tilde{\psi} + \left(\frac{-2(A/B)^{s+1}}{s+1} + \frac{(A/B)^s}{s}\right)\psi + \frac{2(A/B)^s}{s}\left(\frac{C}{B}\psi_1\right).$$

From (1.18) we have $\bar{\partial}\psi(s) = \Phi(s)$ on $\mathfrak{D} - \mathfrak{D}_1$. Let $T(r)$ be the tubular neighborhood of distance r of $\mathfrak{D}_1$ in $\mathfrak{D}$.

$$\int_{\Gamma_1 \backslash \mathfrak{D}} \Phi(s) \wedge \eta = \lim_{\substack{r \to \infty \\ \varepsilon \to 0}} \left[\int_{\partial T(r)} \psi(s) \wedge \eta - \int_{\partial T(\varepsilon)} \psi(s) \wedge \eta\right].$$

Now $\|\psi(s) \wedge \eta\| \prec (A/B)^{\mathrm{Re}(s)}$ and by [11, Proposition (1.10)]

$$\lim_{r \to \infty} \int_{\partial T(r)} \psi(s) \wedge \eta = 0.$$

By [11, Proposition (2.5)]

$$\lim_{\varepsilon \to 0} \int_{\partial T(\varepsilon)} \psi(s) \wedge \eta = \frac{-2}{s(s+1)(s+2)} \int_{\Gamma_1 \backslash \mathfrak{D}_1} \eta.$$

This finishes the proof.

Let Γ be a torsion free discrete subgroup of G and $\Gamma_1 = G_1 \cap \Gamma$. We assume that

$$(1.20) \qquad \Gamma \backslash \mathfrak{D} \quad \text{and} \quad \Gamma_1 \backslash \mathfrak{D}_1$$

are compact. Let π: $\Gamma_1 \backslash \mathfrak{D} \to \Gamma \backslash \mathfrak{D}$ be the projection map. Then its restriction to $\Gamma_1 \backslash \mathfrak{D}_1$ is generically 1-1. In the following, we construct the harmonic dual of the cycle $\pi(\Gamma_1 \backslash \mathfrak{D}_1)$ in $\Gamma \backslash \mathfrak{D}$.

We define

$$(1.21) \qquad \omega(s) = \tfrac{1}{2}s(s+1)(s+2)\Phi(s), \quad \hat{\omega}(S) = \sum_{\Gamma_1 \backslash \Gamma} \gamma^* \omega(s).$$

By formula (1.8), an estimation of $\|\Phi(s)\|$ shows

$$(1.22) \qquad \|\Phi(s)\| = O\left(\left(\frac{A}{B}\right)^{\mathrm{Re}(s)+2}\right).$$

From [11, Proposition (1.13)], the series $\hat{\omega}(s)$ converges absolutely for $\mathrm{Re}(s) > p-1$. It is an immediate consequence of Proposition (1.3) and an unfolding argument that $\hat{\omega}(s)$ is a dual form of the cycle $\pi(\Gamma_1 \backslash \mathfrak{D}_1)$. In the following, we discuss the analytic continuation of $\hat{\omega}(s)$ to obtain the harmonic dual of the cycle $\pi(\Gamma_1 \backslash \mathfrak{D}_1)$. For this purpose, we need an explicit formula of the Laplacian.

We use the formula of the complex Laplacian

$$\square = i\{\bar{\partial}\partial\Lambda - \bar{\partial}\Lambda\partial + \partial\Lambda\bar{\partial} - \Lambda\partial\bar{\partial}\}. \tag{1.23}$$

This is the same as [11, §4] except that to conform with standard sign conventions [13], [14], the Λ operator here differs by a sign from that of [11, §4] and [4]. Thus in local coordinates (cf. [4, p. 112])

$$\begin{aligned} &i\Lambda\Big\{\sum \varphi_{\alpha_1\cdots\alpha_p\bar{\beta}_1\cdots\bar{\beta}_q} dz_{\alpha_1}\wedge\cdots\wedge d\bar{z}_{\bar{\beta}_q}\Big\} \\ &\qquad = \sum g^{\alpha\bar{\beta}}\varphi_{\alpha\alpha_2\cdots\alpha_p\bar{\beta}\bar{\beta}_2\cdots\bar{\beta}_q} dz_{\alpha_2}\wedge\cdots\wedge d\bar{z}_{\bar{\beta}_q}. \end{aligned} \tag{1.24}$$

We shall be computing $\square$ on G_1 invariant forms. The invariance implies that it suffices to compute at $Z_1 = 0$ where $g^{\alpha\bar{\beta}}$ has the particularly simple form (1.8). This makes the computation of $i\Lambda$ fairly straightforward, and by (1.23) the action of $\square$ becomes effectively computable.

We shall use the following notations:

$$w_i = \begin{pmatrix} z_{i1} \\ \vdots \\ z_{ip-1} \end{pmatrix}, \qquad i = 1, 2, \tag{1.25}$$

so that $Z_1 = (w_1 w_2)$ and

$$v_i = z_{ip}, \qquad i = 1, 2, \tag{1.26}$$

so that $v = (v_1, v_2)$ may be used interchangeably with $Z_2 = (z_{1p}, z_{2p})$. We denote by a, b, c, d the G_1 invariant $(1, 1)$ forms whose value at $Z_1 = 0$ are given by

$$\begin{aligned} a &= {}^t dw_1 \wedge d\bar{w}_1 + {}^t dw_2 \wedge d\bar{w}_2 = -\partial\bar{\partial}\log B, \\ b &= \left(\frac{B}{A}\right)\bar{v}\,{}^t dZ_1 \wedge d\bar{Z}_1{}^t v, \\ c &= \left(\frac{B}{A}\right)(dv_1 \wedge d\bar{v}_1 + dv_2 \wedge d\bar{v}_2), \\ d &= \left(\frac{B}{A}\right)^2 \bar{v}\,{}^t dv \wedge d\bar{v}\,{}^t v = \left(\frac{B}{A}\right)^2 \partial\left(\frac{A}{B}\right) \wedge \bar{\partial}\left(\frac{A}{B}\right). \end{aligned} \tag{1.27}$$

The following are easy to check:

$$-\partial\bar\partial \log B = a, \quad -\partial\bar\partial \log A = a + b + c + d,$$

$$(1.28) \qquad \bar Z_2(E - {}^t Z_1 \bar Z_1)^{-1}\, {}^t dZ_1 (E - \bar Z_1 {}^t Z_1)^{-1} d\bar Z_1 (E - {}^t Z_1 \bar Z_1)^{-1}\, {}^t Z_2 = \left(\frac{A}{B}\right) b,$$

$$\partial\left(\frac{A}{B}\right) \wedge \bar\partial\left(\frac{A}{B}\right) = \left(\frac{A}{B}\right)^2 d, \quad -\partial\bar\partial\left(\frac{A}{B}\right) = \frac{A}{B}(b + c).$$

Lemma (1.4).

$$i\Lambda(a) = (p-1)\left(1 + \frac{A}{B}\right),$$

$$i\Lambda(b) = (p-1)\left(1 - \frac{A}{B}\right),$$

$$i\Lambda(c) = 1 + \frac{A}{B},$$

$$i\Lambda(d) = 1 - \frac{A}{B}.$$

Proof. We compute $i\Lambda(b)$; the other ones are easier. At $Z_1 = 0$,

$$\begin{aligned} i\Lambda(b) &= \left(\frac{B}{A}\right) i\Lambda\{|v_1|^2\, {}^t dw_1 \wedge d\bar w_1 + v_1 \bar v_2\, {}^t dw_2 \wedge d\bar w_1 \\ &\qquad + \bar v_1 v_2\, {}^t dw_1 \wedge d\bar w_2 + |v_2|^2\, {}^t dw_2 \wedge d\bar w_2\} \\ &= \frac{B}{A}\{|v_1|^2 (p-1)(1 - |v_1|^2) - v_1 \bar v_2 (p-1) \bar v_1 v_2 \\ &\qquad - \bar v_1 v_2 (p-1) v_1 \bar v_2 + |v_2|^2 (p-1)(1 - |v_2|^2)\} \\ &= (p-1)\frac{B}{A}\{(1 - |Z_2|^2) - (1 - |Z_2|^2)^2\} = (p-1)\left(1 - \frac{A}{B}\right). \end{aligned}$$

Again at $Z_1 = 0$ the curvature matrices Ω, $\tilde\Omega$ take the forms

$$(1.29) \qquad \begin{aligned} \Omega &= -\begin{pmatrix} {}^t dw_1 \wedge d\bar w_1 & {}^t dw_1 \wedge d\bar w_2 \\ {}^t dw_1 \wedge d\bar w_1 & {}^t dw_2 \wedge d\bar w_2 \end{pmatrix}, \\ \tilde\Omega &= -\begin{pmatrix} {}^t dw_1 \wedge d\bar w_1 + \dfrac{dv_1 \wedge d\bar v_1}{A} & {}^t dw_1 \wedge d\bar w_2 + \dfrac{dv_1 \wedge d\bar v_2}{A} \\ {}^t dw_2 \wedge d\bar w_1 + \dfrac{dv_2 \wedge d\bar v_1}{A} & {}^t dw_2 \wedge d\bar w_2 + \dfrac{dv_2 \wedge d\bar v_2}{A} \end{pmatrix} \\ &\quad \cdot (E - {}^t Z_2 \bar Z_2)^{-1}. \end{aligned}$$

Hence we get for the Chern forms:

$$C(E)=\frac{-1}{4\pi^2}\{{}^t dw_1\wedge d\bar w_1\wedge {}^t dw_2\wedge d\bar w_2-{}^t dw_2\wedge d\bar w_1\wedge {}^t dw_1\wedge d\bar w_2\},$$

$$\begin{aligned}\tilde C(E)=\frac{-1}{4\pi^2}\left(\frac{B}{A}\right)\Big\{&{}^t dw_1\wedge d\bar w_1\wedge {}^t dw_2\wedge d\bar w_2-{}^t dw_2\wedge d\bar w_1\wedge {}^t dw_1\wedge d\bar w_2\\ &+\frac{B}{A}\left[{}^t dw_1\wedge d\bar w_1\wedge dv_2\wedge d\bar v_2+{}^t dw_2\wedge d\bar w_2\wedge dv_1\wedge d\bar v_1\right.\\ &\left.-{}^t dw_2\wedge d\bar w_1\wedge dv_1\wedge d\bar v_2-{}^t dw_1\wedge d\bar w_2\wedge dv_2\wedge d\bar v_1\right]\\ &+2\left(\frac{B}{A}\right)^2 dv_1\wedge d\bar v_1\wedge dv_2\wedge d\bar v_2\Big\}\end{aligned}\tag{1.30}$$

and by [11, (2.15)]

$$\begin{aligned}\bar\partial\left(\frac{C}{B}\psi_1\right)=\frac{-1}{8\pi^2}\Big\{&-\left[{}^t dw_1\wedge d\bar w_1\wedge dv_2\wedge d\bar v_2+{}^t dw_2\wedge d\bar w_2\wedge dv_1\right.\\ &\left.\wedge d\bar v_1-{}^t dw_2\wedge d\bar w_1\wedge dv_1\wedge d\bar v_2-{}^t dw_1\wedge d\bar w_2\wedge dv_2\wedge d\bar v_1\right]\\ &-(-4\pi^2)\left(1-\frac{A}{B}\right)C(E)\Big\}.\end{aligned}\tag{1.31}$$

Lemma (1.5). (i) $(-4\pi^2)(i\Lambda)C(E)=pA(a+b)/B$.
(ii) $(-4\pi^2)(i\Lambda)\tilde C(E)=(p+1)(a+b+c+d)$.
(iii) $(-4\pi^2)(i\Lambda)\bar\partial(C\psi_1/B)=\frac{1}{2}\{(-pA/B+(p-1)(A/B)^2)(a+b)+(1-p)(A/B)^2(c+d)\}$.

Proof. We prove (i); the other ones are similar.

$$\begin{aligned}i\Lambda({}^t dw_1\wedge d\bar w_1\wedge {}^t dw_2\wedge d\bar w_2)&=(p-1)(1-|v_1|^2)\,{}^t dw_2\wedge d\bar w_2\\ &+(p-1)(1-|v_2|^2)\,{}^t dw_1\wedge d\bar w_1\\ &+v_1\bar v_2\,{}^t dw_2\wedge d\bar w_1+\bar v_1 v_2\,{}^t dw_1\wedge d\bar w_2,\\ i\Lambda({}^t dw_2\wedge d\bar w_1\wedge {}^t dw_1\wedge d\bar w_2)&=-(p-1)v_1\bar v_2\,{}^t dw_2\wedge d\bar w_1\\ &-(p-1)\bar v_1 v_2\,{}^t dw_1\wedge d\bar w_2-(1-|v_1|^2)\,{}^t dw_2\\ &\wedge d\bar w_2-(1-|v_2|^2)\,{}^t dw_1\wedge d\bar w_1.\end{aligned}$$

From (1.27) and (1.30) the identity (i) follows.

For a G_1 invariant form f with bounded norm we define

$$f_s=\sum_{\Gamma_1\setminus\Gamma}\gamma^*\left(\left(\frac{A}{B}\right)^{s+2}f\right).\tag{1.32}$$

The series is absolutely convergent for $\mathrm{Re}(s) > p-1$ and has at most a single pole at $s = p-1$ because the series $\Sigma_{\Gamma_1\setminus\Gamma}\gamma^*(A/B)^{s+2}$ has such a property. From the formula of $(g^{\alpha\beta})$ one shows readily that $a. b. c. d$ have bounded norm, thus we can consider a_s, b_s, c_s, d_s.

Lemma (1.6).

$$-\Box\hat{\omega}(s) = \left(\frac{-1}{4\pi^2}\right)(s-p+1)\partial\bar{\partial}\{(s+1)(a_s+b_s+c_s+d_s) - (s+2)(a_s+b_s)\}.$$

Proof. By our construction $\omega(s)$ is a real form which is both ∂ and $\bar{\partial}$ closed. Therefore by (1.23)

$$-\Box\omega(s) = \partial\bar{\partial}\{(i\Lambda)\omega(s)\}.$$

By Lemma (1.5) and (1.21), (1.19)

$$(1.33)\quad (i\Lambda)\omega(s) = \left(\frac{-1}{4\pi^2}\right)(s-p+1)\{-(a+b)+(s+1)(c+d)\}\left(\frac{A}{B}\right)^{s+2}.$$

This proves the lemma.

To discuss analytic continuation we need the action of Laplacian on $\partial\bar{\partial}a_s$, etc.

Lemma (1.7). (i) $i\Lambda(a\wedge a) = 2(p-2)a + 2(p-1)(A/B)a + 2(A/B)b$.
(ii) $i\Lambda(b\wedge b) = 2(p-2)(1-A/B)b$.
(iii) $i\Lambda(a\wedge b) = (p-1)(1-A/B)a + ((p-1)+(p-3)A/B)b$.
(iv) $i\Lambda(c\wedge c) = 2(A/B)(c+d)$.
(v) $i\Lambda(c\wedge d) = (1-A/B)(c+d)$.

Proof. We prove (i); the other ones are similar although (iii) is tedious.

$$\begin{aligned} i\Lambda&\Big(({}^t dw_1\wedge d\bar{w}_1)^2 + 2\,{}^t dw_1\wedge d\bar{w}_1\wedge {}^t dw_2\wedge d\bar{w}_2 + ({}^t dw_2\wedge d\bar{w}_2)^2\Big)\\ &= 2(p-2)(1-|v_1|^2)\,{}^t dw_1\wedge d\bar{w}_1\\ &\quad +2(p-1)(1-|v_1|^2)\,{}^t dw_2\wedge d\bar{w}_2 + 2(p-1)(1-|v_2|^2)\,{}^t dw_1\wedge d\bar{w}_1\\ &\quad +2v_1\bar{v}_2\,{}^t dw_2\wedge d\bar{w}_1 + 2\bar{v}_1 v_2\,{}^t dw_1\wedge d\bar{w}_2\\ &\quad +2(p-2)(1-|v_2|^2)\,{}^t dw_2\wedge d\bar{w}_2\\ &= 2(p-2)a + 2(p-1)\left(\frac{A}{B}\right)a + 2\left(\frac{A}{B}\right)b. \end{aligned}$$

Remark. The contractions of $a \wedge c$, $a \wedge d$, $b \wedge c$, $b \wedge d$ follow immediately from Lemma (1.4) since in these products the factors do not have common variables.

Lemma (1.8).

$$\begin{aligned}
&(i\Lambda)\partial\bar\partial\left(\frac{A}{B}\right)^{s+2}(-\partial\bar\partial\log A)\\
&\quad= (s+2)\{(s-p+1)a + (s-3p+3)b + (s-3p+3)c\\
&\qquad + ((2p-1)s + (p-1))d - (s-p+3)\frac{A}{B}(-\partial\bar\partial\log A)\}\left(\frac{A}{B}\right)^{s+2}\\
&\quad= (s+2)\{(s-3p+3)(-\partial\bar\partial\log A) + 2(p-1)a\\
&\qquad +2(p-1)(s+2)d - (s-p+3)\frac{A}{B}(-\partial\bar\partial\log A)\}\left(\frac{A}{B}\right)^{s+2}.
\end{aligned}$$

Proof. We have

$$\begin{aligned}
\partial\bar\partial\left(\frac{A}{B}\right)^{s+2}(-\partial\bar\partial\log A) = (s+2)\left(\frac{A}{B}\right)^{s+2}&\{-(b+c)+(s+1)d\}\\
&\cdot(a+b+c+d).
\end{aligned}$$

Then the result follows from the previous lemma.

Lemma (1.9).

$$\begin{aligned}
&(i\Lambda)\partial\bar\partial\left(\frac{A}{B}\right)^{s+2}(-\partial\bar\partial\log B)\\
&\quad= (s+2)\left(\frac{A}{B}\right)^{s+2}\{(s-p+1)a - (p-1)(b+c) + (s+1)(p-1)d\\
&\qquad + \frac{A}{B}[-(s-p+3)a - (p-3)b - (p-1)c + (s+1)(p-1)d]\}\\
&\quad= (s+2)\left(\frac{A}{B}\right)^{s+2}\{-(p-1)(-\partial\bar\partial\log A) + sa + (p-1)(s+2)d\\
&\qquad - (p-1)\frac{A}{B}(-\partial\bar\partial\log A)\\
&\qquad + \frac{A}{B}[-(s-2p+4)a + 2b + (p-1)(s+2)d]\}.
\end{aligned}$$

Lemma (1.10).

$$(i\Lambda)\partial\bar\partial\left(\left(\frac{A}{B}\right)^{s+2}d\right) = \left(\frac{A}{B}\right)^{s+2}\Big\{(s-2p+2)(b+c)+spd$$
$$+\frac{A}{B}[-(s-2p+6)b-(s-2p+4)c-(sp+2)d]\Big\}$$
$$=\left(\frac{A}{B}\right)^{s+2}\Big\{(s-2p+2)(-\partial\bar\partial\log A)-(s-2p+2)aD$$
$$+(s+2)(p-1)d-(s-2p+4)\frac{A}{B}(-\partial\bar\partial\log A)$$
$$+\frac{A}{B}[(s-2p+4)a-2b-(s+2)(p-1)d]\Big\}$$

From (1.28) we have

$$\left(\frac{A}{B}\right)^{s+2}b=\left(\frac{A}{B}\right)^{s+1}\bar Z_2\left(E-{}^t Z_1\bar Z_1\right)^{-1}{}^t dZ_1\left(E-\bar Z_1{}^t Z_1\right)^{-1}d\bar Z_1\left(E-{}^t Z_1\bar Z_1\right)^{-1}{}^t Z_2$$

and thus at $Z_1=0$

$$\partial\bar\partial\left(\left(\frac{A}{B}\right)^{s+2}b\right)=\left(\partial\bar\partial\left(\frac{A}{B}\right)^{s+1}\right)\bar v{}^t dZ_1\wedge d\bar Z_1{}^t v$$
$$+\left(\frac{A}{B}\right)^{s+1}\bar v{}^t dZ_1\wedge d\bar Z_1\wedge{}^t dZ_1\wedge d\bar Z_1{}^t v$$
$$(1.34)\qquad -\left(\frac{A}{B}\right)^{s+1}d\bar v\wedge{}^t dZ_1\wedge d\bar Z_1{}^t dv$$
$$+(s+1)\left(\frac{A}{B}\right)^{s}\Big\{\partial\left(\frac{A}{B}\right)d\bar v\wedge{}^t dZ_1\wedge d\bar Z_1{}^t v$$
$$-\bar\partial\left(\frac{A}{B}\right)\bar v{}^t dZ_1\wedge d\bar Z_1{}^t dv\Big\}.$$

From this we have the following formula.

Lemma (1.11).

$$(i\Lambda)\left(\partial\bar\partial\left(\left(\frac{A}{B}\right)^{s+2}b\right)\right)=\left(\frac{A}{B}\right)^{s+2}\Big\{(s^2-2(p-2)s-2(p-1))b$$
$$-(p-1)sc+(p-1)(s+1)sd$$
$$+\frac{A}{B}\big[a-(s^2-2(p-4)s-(4p-11))b$$
$$+(p-1)(s+1)c-(p-1)(s^2+3s+3)d\big]\Big\}$$

$$= \left(\frac{A}{B}\right)^{s+2}\Big\{-(p-1)s(-\partial\bar\partial\log A)+(p-1)sa$$
$$+\big(s^2-(p-3)s-2(p-1)\big)b+(p-1)s(s+2)d$$
$$+(p-1)(s+1)\frac{A}{B}(-\partial\bar\partial\log A)+\frac{A}{B}\Big[-\big((p-1)s+p-2\big)a$$
$$-\big(s^2-(p-7)s-(3p-10)\big)b-(p-1)(s+2)^2d\Big]\Big\}.$$

Now denote

$$\alpha_s=\partial\bar\partial\left(\frac{A}{B}\right)^{s+2}(-\partial\bar\partial\log A),$$
$$\beta_s=\partial\bar\partial\left(\frac{A}{B}\right)^{s+2}(-\partial\bar\partial\log B),$$
$$\gamma_s=\partial\bar\partial\left(\left(\frac{A}{B}\right)^{s+2}b\right),$$
$$\delta_s=\partial\bar\partial\left(\left(\frac{A}{B}\right)^{s+2}d\right), \tag{1.35}$$
$$\hat\alpha_s=\sum_{\Gamma_1\setminus\Gamma}\gamma^*\alpha_s,\quad \hat\beta_s=\sum_{\Gamma_1\setminus\Gamma}\gamma^*\beta_s,$$
$$\hat\gamma_s=\sum_{\Gamma_1\setminus\Gamma}\gamma^*\gamma_s,\quad \hat\delta_s=\sum_{\Gamma_1\setminus\Gamma}\gamma^*\delta_s.$$

These forms are holomorphic for $\mathrm{Re}(s)>p-1$. Now we show that they are holomorphic at $s=p-1$ by analytic continuation. From Lemmas (1.8) to (1.11), the following proposition is immediate.

Proposition (1.12).

$$-\Box\begin{pmatrix}\hat\alpha_s\\ \hat\beta_s\\ \hat\gamma_s\\ \hat\delta_s\end{pmatrix}=M(s)\begin{pmatrix}\hat\alpha_s\\ \hat\beta_s\\ \hat\gamma_s\\ \hat\delta_s\end{pmatrix}+N(s)\begin{pmatrix}\hat\alpha_{s+1}\\ \hat\beta_{s+1}\\ \hat\gamma_{s+1}\\ \hat\delta_{s+1}\end{pmatrix}$$

where

$$M(s)=\begin{pmatrix}(s+2)(s-3p+3) & 2(s+2)(p-1) & 0 & 2(p-1)(s+2)^2\\ -(s+2)(p-1) & (s+2)s & 0 & (p-1)(s+2)^2\\ -(p-1)s & (p-1)s & (s+2)(s-p+1) & (p-1)s(s+2)\\ s-2p+2 & -(s-2p-2) & 0 & (s+2)(p-1)\end{pmatrix}$$

and

$$N(s)=\begin{pmatrix} -(s+2)(s-p+3) & 0 & 0 & 0 \\ -(p-1)(s+2) & -(s+2)(s-2p+4) & 2(s+2) & (p-1)(s+2)^2 \\ (p-1)(s+1) & -((p-1)s+p-2) & -(s^2-(p-7)s-(3p-10)) & -(p-1)(s+2)^2 \\ -(s-2p+4) & s-2p+4 & -2 & -(p-1)(s+2)^2 \end{pmatrix}.$$

To obtain a simpler matrix equation, we make a change of basis. Let

$$(1.36) \qquad \hat{\sigma}_s = \hat{\alpha}_s - \hat{\beta}_s, \quad \hat{\tau}_s = -\hat{\alpha}_s + 2\hat{\beta}_s,$$

$$L(s) = \begin{pmatrix} 1 & -1 & 0 & 0 \\ -1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} M(s) \begin{pmatrix} 2 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

$$T(s) = \begin{pmatrix} 1 & -1 & 0 & 0 \\ -1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} N(s) \begin{pmatrix} 2 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Then we have

$$(1.37) \qquad -\Box \begin{pmatrix} \hat{\sigma}_s \\ \hat{\tau}_s \\ \hat{\gamma}_s \\ \hat{\delta}_s \end{pmatrix} = L(s) \begin{pmatrix} \hat{\sigma}_s \\ \hat{\tau}_s \\ \hat{\gamma}_s \\ \hat{\delta}_s \end{pmatrix} + T(s) \begin{pmatrix} \hat{\sigma}_{s+1} \\ \hat{\tau}_{s+1} \\ \hat{\gamma}_{s+1} \\ \hat{\delta}_{s+1} \end{pmatrix},$$

where

$$L(s)=\begin{pmatrix} (s+2)(s-2p+2) & 0 & 0 & (p-1)(s+2)^2 \\ 0 & (s+2)(s-p+1) & 0 & 0 \\ -(p-1)s & 0 & (s+2)(s-p+1) & (s+2)s(p-1) \\ s-2p+2 & 0 & 0 & (s+2)(p-1) \end{pmatrix}.$$

Lemma (1.13). $\det(L(s)+\lambda E) \neq 0$, *for* $\lambda > 0$ *and* $\mathrm{Re}(s) \geqslant p-1$.

Proof. $\det(L(s)+\lambda E) = ((s+2)(s-p+1)+\lambda)^3\lambda \neq 0$ for $\lambda > 0$ and $\mathrm{Re}(s) \geqslant p-1$.

Proposition (1.14). *As functions of the complex variable s the differential forms* $\partial\bar{\partial}(a_s)$, $\partial\bar{\partial}(b_s)$, $\partial\bar{\partial}(c_s)$, $\partial\bar{\partial}(d_s)$ *have meromorphic continuations to the entire plane and their continuations are regular without poles at the point* $s = p-1$.

Proof. Since $\partial\bar{\partial}(a_s)$, $\partial\bar{\partial}(b_s)$, $\partial\bar{\partial}(c_s)$, $\partial\bar{\partial}(d_s)$ are linear combinations of $\hat{\sigma}_s$, $\hat{\tau}_s$, $\hat{\gamma}_s$, $\hat{\delta}_s$ with constant coefficients, it is enough to verify the assertion for $\hat{\sigma}_s$, $\hat{\tau}_s$, $\hat{\gamma}_s$, $\hat{\delta}_s$. Let η_n $(n = 1, 2, \cdots)$ be an orthonormal basis of eigenforms of degree $(2,2)$ for $\Box$ on $\Gamma\backslash\mathfrak{D}$ and λ_n the corresponding sequence of increasing eigenvalues. Let

$$\hat{\sigma}_s(n) = \langle \hat{\sigma}_s, \eta_n \rangle, \qquad \hat{\tau}_s(n) = \langle \hat{\tau}_s, \eta_n \rangle,$$
$$\hat{\gamma}_s(n) = \langle \hat{\gamma}_s, \eta_n \rangle, \qquad \hat{\delta}_s(n) = \langle \hat{\delta}_s, \eta_n \rangle,$$

where $\langle \hat{\sigma}_s, \eta_n \rangle$ denotes the Hodge inner product. If $\lambda_n = 0$, η_n is harmonic and since $\hat{\sigma}_s$, $\hat{\tau}_s$, $\hat{\gamma}_s$, $\hat{\delta}_s$ are exact, we have $\hat{\sigma}_s(n) = \hat{\tau}_s(n) = \hat{\gamma}_s(n) = \hat{\delta}_s(n) = 0$. Now suppose that $\lambda_n > 0$. Then by formula (1.37) and Lemma (1.13),

$$(1.38) \qquad \begin{pmatrix} \hat{\sigma}_s(n) \\ \hat{\tau}_s(n) \\ \hat{\gamma}_s(n) \\ \hat{\delta}_s(n) \end{pmatrix} = \frac{-\operatorname{adj}(L(s) + \lambda_n E)T(s)}{\lambda_n(\lambda_n + (s+2)(s-p+1))^3} \begin{pmatrix} \hat{\sigma}_{s+1}(n) \\ \hat{\tau}_{s+1}(n) \\ \hat{\gamma}_{s+1}(n) \\ \hat{\delta}_{s+1}(n) \end{pmatrix}.$$

Note $\operatorname{adj}(L(s) + \lambda_n E)$ is polynomial in λ_n of degree $\leqslant 3$. Applying $\Box^l$, we obtain

$$\begin{pmatrix} \langle \Box^l \hat{\sigma}_s, \eta_n \rangle \\ \langle \Box^l \hat{\tau}_s, \eta_n \rangle \\ \langle \Box^l \hat{\gamma}_s, \eta_n \rangle \\ \langle \Box^l \hat{\delta}_s, \eta_n \rangle \end{pmatrix} = \lambda_n^l \begin{pmatrix} \langle \hat{\sigma}_s, \eta_n \rangle \\ \langle \hat{\tau}_s, \eta_n \rangle \\ \langle \hat{\gamma}_s, \eta_n \rangle \\ \langle \hat{\delta}_s, \eta_n \rangle \end{pmatrix},$$

and therefore

$$|\hat{\sigma}_s(n)|, |\hat{\tau}_s(n)|, |\hat{\gamma}_s(n)|, |\hat{\delta}_s(n)|$$
$$\leqslant \max \frac{1}{\lambda_n^l} \{ |\langle \Box^l \hat{\sigma}_s, \eta_n \rangle|, |\langle \Box^l \hat{\tau}_s, \eta_n \rangle|, |\langle \Box^l \hat{\gamma}_s, \eta_n \rangle|, |\langle \Box^l \hat{\delta}_s, \eta_n \rangle| \}.$$

Now since $\lambda_n \sim \text{const.}\, n^{1/2p}$ [2], given any $m > 0$ we can, by choosing l large enough, find constant c_m such that

$$\max\{ |\hat{\sigma}_s(n)|, |\hat{\tau}_s(n)|, |\hat{\gamma}_s(n)|, \hat{\delta}_s(n) \} \leqslant \frac{c_m}{n^m}.$$

This holds for $\operatorname{Re}(s) > p - 1$ and by (1.38),

$$\max\{ |\hat{\sigma}_s(n)|, |\hat{\tau}_s(n)|, |\hat{\gamma}_s(n)|, \hat{\delta}_s(n) \} \leqslant \frac{c'_m}{n^m}$$

for s in the complement of a disjoint union of small open disks around the poles of $\hat{\sigma}_s(n)$, $\hat{\tau}_s(n)$, $\hat{\gamma}_s(n)$, $\hat{\delta}_s(n)$. Thus we obtain a meromorphic continuation of these functions to the entire plane, and by Lemma (1.13) they are holomorphic at $s = p - 1$.

Theorem (1.15). *$\hat{\omega}(s)$ has a meromorphic continuation in s to the entire plane. The continuation is regular at $s = p - 1$, and $\hat{\omega}(p-1)$ is the harmonic form Poincaré dual to $\pi(\Gamma_1 \backslash \mathfrak{D}_1)$.*

Proof. By Lemma (1.6)

$$(1.39) \quad -\square\hat{\omega}(s) = \left(\frac{-1}{4\pi^2}\right)(s-p+1)\{(s+1)\partial\bar{\partial}(a_s+b_s+c_s+d_s) \\ -(s+2)\partial\bar{\partial}(a_s+b_s)\}.$$

By the preceding proposition $\partial\bar{\partial}a_s$, etc. are meromorphic in s. By a similar argument as in the preceding discussion (cf. proof of [11, Theorem (4.10)]) $\hat{\omega}(s)$ is meromorphic. Since $\partial\bar{\partial}a_s$, etc. are regular at $s = p - 1$ it follows that $\hat{\omega}(p-1)$ is regular. Then by (1.39) $\square\hat{\omega}(p-1) = 0$, i.e., $\hat{\omega}(p-1)$ is harmonic, and by our construction $\hat{\omega}(s)$ always has the cohomology class which is the Poincaré dual of $\pi(\Gamma_1 \backslash \mathfrak{D}_1)$.

2. Polynomials as special values of the dual form

In this section we shall identify $\omega(p-1)$ with certain spherical harmonic polynomials. To do this, the differential forms on $\mathfrak{D}$ will be pulled back to a vector space.

Let $n = p + q$, V an n-dimensional complex vector space, and $W = V^q = V \oplus \cdots \oplus V$ (q copies). We identify V with $M_{n1}(\mathbf{C})$ and W with $M_{nq}(\mathbf{C})$. For $X, Y \in W$ or V we define

$$(2.1) \quad \begin{aligned} &\text{(i)} \quad (X, Y) = {}^t\bar{X}E_{p,q}Y, \\ &\text{(ii)} \quad \langle X, Y\rangle = {}^tXE_{p,q}\bar{Y}, \end{aligned}$$

where

$$E_{p,q} = \begin{pmatrix} E_p & 0 \\ 0 & -E_q \end{pmatrix}.$$

We shall also use the notations (2.1) when say $X \in W$ and $Y \in V$. The product $\langle\, ,\rangle$ in the sense (2.1)(ii) will only be used in this section.

Let $W_- = \{X \in W \mid (X, X) < 0\}$ and $\tilde{\mathfrak{D}} = \{X \in W \mid (X, X) = -E_q\}$. For $X \in W$ we write

$$(2.2) \quad X = \begin{pmatrix} X_+ \\ X_- \end{pmatrix},$$

with $X_+ \in M_{pq}(\mathbf{C})$, $X_- \in M_{qq}(\mathbf{C})$. If $X \in W_-$, it is easy to see that X_- is invertible. Thus we have a map $\pi: W_- \to \mathfrak{D}$ given by

$$\pi(X) = X_+ X_-^{-1}. \tag{2.3}$$

Let G denote $SU(p, q)$. Clearly π is holomorphic and G-equivariant with G acting on W by left translation. Although $\tilde{\mathfrak{D}}$ is not a complex submanifold of W_-, one can select canonically for every $X \in \tilde{\mathfrak{D}}$ a complex subspace of $T_X(\tilde{\mathfrak{D}})$ which identifies with $T_{\pi(X)}(\mathfrak{D})$ under π_*. For $X \in W$, we define

$$X^\perp = \{Y \in W \mid (X, Y) = 0\}. \tag{2.4}$$

The following lemma is straightforward.

Lemma (2.1). *For $X \in \tilde{\mathfrak{D}}$, $X^\perp$ is a complex subspace of $T_X(W_-)$ contained in $T_X(\tilde{\mathfrak{D}})$ such that*

(i) *π_* identifies $X^\perp$ with $T_{\pi(X)}(\mathfrak{D})$,*

(ii) *for every $g \in G$,*

$$g_*(X^\perp) = (gX)^\perp.$$

In the rest of this section again assume $q = 2$. For $M \in V$ with $(M, M) > 0$, we write $\langle M \rangle$ for the span of M and $G_{\langle M \rangle}$ for the isotropic subgroup of G at the line $\langle M \rangle$. We define a $G_{\langle M \rangle}$ action on the trivial bundle $\mathfrak{D} \times (\mathbf{C}^2 \otimes \langle M \rangle)$. For $g \in G_{\langle M \rangle}$, $(Z, \binom{a}{b} \otimes X) \in \mathfrak{D} \times (\mathbf{C}^2 \otimes \langle M \rangle)$,

$$g(Z, \tbinom{a}{b} \otimes X) = \left(gZ, {}^t j(g, Z)^{-1} \tbinom{a}{b} \otimes gX\right). \tag{2.5}$$

Let $e_1, \cdots, e_n$ be the standard basis of V and $G_1 = G_{\langle e_p \rangle}$. Identifying $\mathbf{C}^2 \otimes \langle e_p \rangle$ with $\mathbf{C}^2$, clearly the action (2.5) coincides with that of (1.12). More generally for $h \in G$ the same formula (2.5) defines a map $h: \mathfrak{D} \times (\mathbf{C}^2 \otimes \langle M \rangle) \to \mathfrak{D} \times (\mathbf{C}^2 \otimes \langle hM \rangle)$, and we have a commutative diagram

$$\begin{array}{ccc} \mathfrak{D} \times (\mathbf{C}^2 \otimes \langle M \rangle) & \xrightarrow{h} & \mathfrak{D} \times (\mathbf{C}^2 \otimes \langle hM \rangle) \\ \downarrow g & & \downarrow hgh^{-1} \\ \mathfrak{D} \times (\mathbf{C}^2 \otimes \langle M \rangle) & \xrightarrow{h} & \mathfrak{D} \times (\mathbf{C}^2 \otimes \langle hM \rangle) \end{array} \tag{2.6}$$

with $g \in G_{\langle M \rangle}$.

For $M \in V$ with $(M, M) > 0$, by [11, Definition 1.1] there is a totally geodesic subdomain $\mathfrak{D}_{\langle M \rangle}$. Let $\mathfrak{D}_1$ be as in (1.4) and $g \in G$ with $g_{\langle e_p \rangle} = \langle M \rangle$, then $\mathfrak{D}_{\langle M \rangle} = g\mathfrak{D}_1$. The dual form for the cycle $\Gamma_1 \diagdown \mathfrak{D}_1$ is constructed from the two Hermitian fiber metrics $(E - {}^t Z\bar{Z})^{-1}$ and $(E - {}^t Z_1 \bar{Z}_1)^{-1}$. Now we derive these forms for $\mathfrak{D}_{\langle M \rangle}$. We fix an M and identify the fiber $\mathbf{C}^2 \otimes \langle M \rangle$ with $\mathbf{C}^2$ via

$$\tbinom{a}{b} \otimes \lambda M \mapsto \tbinom{\lambda a}{\lambda b}.$$

Lemma (2.2). *Let $g \in G$ such that $g\langle e_p \rangle = \langle M \rangle$:*

(i) $(g^{-1})^*(E - {}^tZ\bar{Z})^{-1} = (M, M)(E - {}^tZ\bar{Z})^{-1}$,

(ii) $(g^{-1})^*(E - {}^tZ_1\bar{Z}_1)^{-1} = (M, M)\left\{E - {}^tZ\bar{Z} + \dfrac{{}^t(M, \binom{Z}{E})\overline{(M, \binom{Z}{E})}}{(M, M)}\right\}^{-1}$.

Proof. (i) Let $X = \lambda M$, by (2.5)

$$\begin{aligned}
&(g^{-1})^*(E - {}^tZ\bar{Z})^{-1}((\tbinom{a}{b}) \otimes X, (\tbinom{a}{b}) \otimes X) \\
&= (\bar{a}\bar{b}) j^{-1}\overline{(g^{-1}, Z)}\,(E - {}^tg^{-1}Z\,\overline{g^{-1}Z})^t j^{-1}(g^{-1}, Z)(\tbinom{a}{b})(g^{-1}X, g^{-1}X) \\
&= |\lambda|^2 (M, M)(\bar{a}\bar{b})(E - {}^tZ\bar{Z})^{-1}(\tbinom{a}{b}) \\
&= (M, M)(E - {}^tZ\bar{Z})^{-1}((\tbinom{a}{b}) \otimes X, (\tbinom{a}{b}) \otimes X).
\end{aligned}$$

The factor (M, M) arises because of different identifications of $\mathbf{C}^2 \otimes \langle e_p \rangle$ and $\mathbf{C}^2 \otimes \langle M \rangle$ with $\mathbf{C}^2$.

(ii) $E - {}^tZ_1\bar{Z}_1 = E - {}^tZ\bar{Z} + {}^t(e_p, \binom{Z}{E})\overline{(e_p, \binom{Z}{E})}$. It follows that

$$E - {}^t(g^{-1}Z)_1\overline{(g^{-1}Z)}_1 = {}^tj^{-1}(g^{-1}, Z)\left\{E - {}^tZ\bar{Z} + \frac{{}^t(M, \binom{Z}{E})\overline{(M, \binom{Z}{E})}}{(M, M)}\right\} \cdot j^{-1}\overline{(g^{-1}, Z)}$$

and the proof proceeds as in (i).

For $X \in W$, we write $X_{M^\perp}$ for the component of X which is orthogonal to M with respect to inner (2.1) (cf. the remark there). In terms of matrix multiplication,

$$X = M\frac{(M, X)}{(M, M)} + X_{M^\perp}. \tag{2.7}$$

Now we pull back the data to W_- via π: $W_- \to \mathfrak{D}$.

Lemma (2.3). (i) $\pi^*(E - {}^tZ\bar{Z})^{-1} = -\bar{X}_-\langle X, X\rangle^{-1}\,{}^tX_-$,

(ii) $\pi^*\left\{E - {}^tZ\bar{Z} + \dfrac{{}^t(M, \binom{Z}{E})\overline{(M, \binom{Z}{E})}}{(M, M)}\right\}^{-1} = -\bar{X}_-\langle X_{M^\perp}, X_{M^\perp}\rangle^{-1}\,{}^tX_-$

$(X \in W_-)$.

Proof. (i) follows from

$$\begin{aligned}
E - {}^t\pi(X)\overline{\pi(X)} &= -{}^t\begin{pmatrix}\pi(X)\\ E\end{pmatrix} E_{p,q}\begin{pmatrix}\overline{\pi(X)}\\ E\end{pmatrix} \\
&= -{}^tX_-^{-1}\,{}^tXE_{p,q}\bar{X}\bar{X}_-^{-1} = -{}^tX_-^{-1}\langle X, X\rangle\bar{X}_-^{-1}.
\end{aligned}$$

(ii) follows from

$$E - {}^t\pi(X)\overline{\pi(X)} + \frac{{}^t\left(M, \binom{\pi(X)}{E}\right)\overline{\left(M, \binom{\pi(X)}{E}\right)}}{(M, M)}$$
$$= -{}^tX_-^{-1}\left\{\langle X, X\rangle - \frac{{}^t(M, X)\overline{(M, X)}}{(M, M)}\right\}\overline{X}_-^{-1}$$

and $\langle X, X\rangle = \langle X_{M^\perp}, X_{M^\perp}\rangle + {}^t(M, X)\overline{(M, X)}/(M, M)$ by (2.7).

Recall that $B/A = \cosh^2 d(Z, \mathfrak{D}_1)$. Choose $g \in G$ such that $g\langle e_p\rangle = \langle M\rangle$ as above. We have

$$d\left(Z, \mathfrak{D}_{\langle M\rangle}\right) = d\left(Z, g\mathfrak{D}_1\right) = d\left(g^{-1}Z, \mathfrak{D}_1\right) = (g^{-1})^*\left(d(Z, \mathfrak{D}_1)\right).$$

We denote

$$\left(\frac{B}{A}\right)_{\langle M\rangle} = (g^{-1})^*\left(\frac{B}{A}\right); \tag{2.8}$$

then

$$\left(\frac{B}{A}\right)_{\langle M\rangle} = \cosh^2 d\left(Z, \mathfrak{D}_{\langle M\rangle}\right). \tag{2.9}$$

Lemma (2.4). *Let $X \in W_-$. Then*

$$\pi^*\left(\left(\frac{B}{A}\right)_{\langle M\rangle}\right)(X) = \frac{\det\langle X_{M^\perp}, X_{M^\perp}\rangle}{\det\langle X, X\rangle} = \frac{\det(M_{X^\perp}, M_{X^\perp})}{(M, M)}$$
$$= 1 - \frac{(M, X)(X, X)^{-1}(X, M)}{(M, M)}.$$

Proof.

$$\pi^*\left(\left(\frac{B}{A}\right)_{\langle M\rangle}\right)(X) = \pi^*\left((g^{-1})^*\left(\frac{B}{A}\right)\right)(X) = \frac{\det\langle X_{M^\perp}, X_{M^\perp}\rangle}{\det\langle X, X\rangle}$$

by Lemmas (2.2) and (2.3). Next,

$$E - {}^tZ_1\overline{Z}_1 = E - {}^tZ\overline{Z} + {}^tZ_2\overline{Z}_2$$
$$= \left(E - {}^tZ\overline{Z}\right)^{1/2}\left\{E + \left(E - {}^tZ\overline{Z}\right)^{-1/2}{}^tZ_2\overline{Z}_2\left(E - {}^tZ\overline{Z}\right)^{-1/2}\right\}\left(E - {}^tZ\overline{Z}\right)^{1/2},$$

thus

$$\frac{B}{A} = \det\left(E + \left(E - {}^tZ\overline{Z}\right)^{-1/2}{}^tZ_2\overline{Z}_2\left(E - {}^tZ\overline{Z}\right)^{-1/2}\right)$$
$$= 1 + \overline{Z}_2\left(E - {}^tZ\overline{Z}\right)^{-1}{}^tZ_2$$
$$= 1 + \overline{\left(e_p, \binom{Z}{E}\right)}\left(E - {}^tZ\overline{Z}\right)^{-1}{}^t\left(e_p, \binom{Z}{E}\right).$$

It follows that

$$\pi^*\left(\left(\frac{B}{A}\right)_{\langle M\rangle}\right)(X) = 1 - \frac{\overline{(M, X)}\,\langle X, X\rangle^{-1}\,{}^t(X, M)}{(M, M)}$$
$$= 1 - \frac{(M, X)(X, X)^{-1}(X, M)}{(M, M)}.$$

Finally since $M = X(X, X)^{-1}(X, M) + M_{X^\perp}$,

$$(M, M) = (M, X)(X, X)^{-1}(X, M) + (M_{X^\perp}, M_{X^\perp}),$$
$$1 - \frac{(M, X)(X, X)^{-1}(X, M)}{(M, M)} = \frac{(M_{X^\perp}, M_{X^\perp})}{(M, M)}.$$

We now proceed to pull back $\Phi(p-1)$ where $\Phi(s)$ is defined in (1.19). For simplicity of notation we omit the pull back π^* in the following discussion. Let $M \in V$ with $(M, M) > 0$. Corresponding to the cycle of the image of $\mathfrak{D}_{\langle M\rangle}$, we have by Lemmas (2.2) and (2.3) the fiber metrics

$$\tilde{H}_X = -\bar{X}_-\langle X, X\rangle^{-1}\,{}^tX_-,$$
$$H_X = -\bar{X}_-\langle X_{M^\perp}, X_{M^\perp}\rangle^{-1}\,{}^tX_- \qquad (X \in W_-). \tag{2.10}$$

From (2.10), one can compute the connection and curvature matrices. Here we have

$$\langle X_{M^\perp}, X_{M^\perp}\rangle = \langle X, X\rangle - \frac{\langle X, M\rangle\langle M, X\rangle}{\langle M, M\rangle},$$

$$\tilde{\omega} = \tilde{H}^{-1}\partial\tilde{H} = -{}^tX_-^{-1}\langle dX, X\rangle\langle X, X\rangle^{-1}\,{}^tX_- + {}^tX_-^{-1}d\,{}^tX_-,$$
$$\tilde{\Omega} = \bar{\partial}(\tilde{\omega})$$
$$= {}^tX_-^{-1}\{\langle dX, dX\rangle - \langle dX, X\rangle\langle X, X\rangle^{-1}\langle X, dX\rangle\}\langle X, X\rangle^{-1}\,{}^tX_-,$$

(2.11)

$$\omega = H^{-1}\partial H = {}^tX_-^{-1}\left\{\frac{\langle dX, M\rangle\langle M, X\rangle}{\langle M, M\rangle} - \langle dX, X\rangle\right\}\langle X_{M^\perp}, X_{M^\perp}\rangle^{-1}\,{}^tX_- + {}^tX_-^{-1}d\,{}^tX_-,$$

$$\Omega = {}^tX_-^{-1}\left\{\langle dX, dX\rangle - \frac{\langle dX, M\rangle\langle M, dX\rangle}{\langle M, M\rangle}\right\}\langle X_{M^\perp}, X_{M^\perp}\rangle^{-1}\,{}^tX_-$$
$$- {}^tX_-^{-1}\left\{\langle dX, X\rangle - \frac{\langle dX, M\rangle\langle M, X\rangle}{\langle M, M\rangle}\right\}\langle X_{M^\perp}, X_{M^\perp}\rangle^{-1}$$
$$\cdot\left\{\langle X, dX\rangle - \frac{\langle X, M\rangle\langle M, dX\rangle}{\langle M, M\rangle}\right\}\langle X_{M^\perp}, X_{M^\perp}\rangle^{-1}\,{}^tX_-.$$

The formulas in (2.11) simplify considerably when we restrict to $X \in \tilde{\mathfrak{D}}$ and the differentials to the subtangent space $X^{\perp}$. By Lemma (2.1) our forms on $\mathfrak{D}$ identify with this restriction. Note that for $X \in \tilde{\mathfrak{D}}$, $(X, X) = -E$ and on $X^{\perp}$ both $(dX, X) = (X, dX) = 0$.

Lemma (2.5). *Restricting to $X \in \tilde{\mathfrak{D}}$ and to $X^{\perp}$, the connection and curvature matrices are given by*

$$\begin{aligned}
\tilde{\omega} &= {}^tX_-^{-1} d\,{}^tX_-,\\
\tilde{\Omega} &= {}^tX_-^{-1} \langle dX, dX\rangle\, {}^tX_-,\\
\omega &= {}^tX_-^{-1} d\,{}^tX_- + {}^tX_-^{-1} \frac{\langle dX, M\rangle\langle M, X\rangle}{\langle M, M\rangle} \langle X_{M^\perp}, X_{M^\perp}\rangle^{-1}\, {}^tX_-,\\
\Omega &= {}^tX_-^{-1}\left\{\langle dX, dX\rangle - \frac{\langle dX, M\rangle\langle M, dX\rangle}{\langle M, M\rangle}\left(\frac{A}{B}\right)_{\langle M\rangle}\right\}\\
&\quad\cdot \langle X_{M^\perp}, X_{M^\perp}\rangle^{-1}\, {}^tX_-.
\end{aligned}$$

Proof. The expressions for $\tilde{\omega}$, $\tilde{\Omega}$ and ω are immediate from (2.11), and only the formula for Ω needs justification. From (2.11), we have

$$\begin{aligned}
\Omega = {}&{}^tX_-^{-1} \langle dX, dX\rangle \langle X_{M^\perp}, X_{M^\perp}\rangle^{-1}\, {}^tX_-\\
&- {}^tX_-^{-1} \frac{\langle dX, M\rangle}{\langle M, M\rangle^{1/2}}\left\{1 + \frac{\langle M, X\rangle}{\langle M, M\rangle^{1/2}} \langle X_{M^\perp}, X_{M^\perp}\rangle^{-1} \frac{\langle X, M\rangle}{\langle M, M\rangle^{1/2}}\right\}\\
&\cdot \frac{\langle M, dX\rangle}{\langle M, M\rangle^{1/2}} \langle X_{M^\perp}, X_{M^\perp}\rangle^{-1}\, {}^tX_-.
\end{aligned}$$

However, we have the identities

$$\begin{aligned}
1 &+ \frac{\langle M, X\rangle}{\langle M, M\rangle^{1/2}} \langle X_{M^\perp}, X_{M^\perp}\rangle^{-1} \frac{\langle X, M\rangle}{\langle M, M\rangle^{1/2}}\\
&= \det\left(E + \langle X_{M^\perp}, X_{M^\perp}\rangle^{-1/2} \frac{\langle X, M\rangle\langle M, X\rangle}{\langle M, M\rangle} \langle X_{M^\perp}, X_{M^\perp}\rangle^{-1/2}\right)\\
&= \det\langle X_{M^\perp}, X_{M^\perp}\rangle^{-1} \det\left(\langle X_{M^\perp}, X_{M^\perp}\rangle + \frac{\langle X, M\rangle\langle M, X\rangle}{\langle M, M\rangle}\right)\\
&= \frac{\det\langle X, X\rangle}{\det\langle X_{M^\perp}, X_{M^\perp}\rangle} = \left(\frac{A}{B}\right)_{\langle M\rangle} \quad \text{(by Lemma (2.4))}.
\end{aligned} \tag{2.12}$$

This proves the formula for Ω.

The space $\tilde{\mathfrak{D}}$ is contained in $M_{n2}(\mathbf{C})$. For $X \in M_{n2}(\mathbf{C})$, we write $X = (X_1 X_2)$ where X_i is the i-th column of X. From Lemma (2.5), we have the following explicit formulas for Chern classes:

$$(-4\pi^2)\tilde{C}(E) = \det\langle dX, dX\rangle,$$

$$(-4\pi^2)C(E) = \left(\frac{A}{B}\right)_{\langle M\rangle}\det\left\{\langle dX, dX\rangle - \left(\frac{A}{B}\right)_{\langle M\rangle}\frac{\langle dX, M\rangle\langle M, dX\rangle}{\langle M, M\rangle}\right\}$$

$$(2.13)\qquad = \left(\frac{A}{B}\right)_{\langle M\rangle}\left\{\det\langle dX, dX\rangle - \left(\frac{A}{B}\right)_{\langle M\rangle}\begin{vmatrix}\langle dX_1, dX_1\rangle & \dfrac{\langle dX_1, M\rangle\langle M, dX_2\rangle}{\langle M, M\rangle}\\ \langle dX_2, dX_1\rangle & \dfrac{\langle dX_2, M\rangle\langle M, dX_2\rangle}{\langle M, M\rangle}\end{vmatrix}\right.$$

$$\left. - \left(\frac{A}{B}\right)_{\langle M\rangle}\begin{vmatrix}\dfrac{\langle dX_1, M\rangle\langle M, dX_1\rangle}{\langle M, M\rangle} & \langle dX_1, dX_2\rangle\\ \dfrac{\langle dX_2, M\rangle\langle M, dX_1\rangle}{\langle M, M\rangle} & \langle dX_2, dX_2\rangle\end{vmatrix} + \left(\frac{A}{B}\right)^2_{\langle M\rangle}\det\frac{\langle dX, M\rangle\langle M, dX\rangle}{\langle M, M\rangle}\right\}.$$

To compute $\Phi(p-1)$, it remains to evaluate $\bar{\partial}(C\psi_1/B)$. For this purpose, we examine K, $\tilde{K}$ and $\alpha = dv d\bar{v}$. By (1.15), we have

$$(2.14)\qquad \begin{aligned}\tilde{K} &= e^t X_-^{-1}\langle dX, dX\rangle \bar{X}_-^{-1}\,{}^t\bar{e},\\ K &= e^t X_-^{-1}\left\{\langle dX, dX\rangle - \frac{\langle dX, M\rangle\langle M, dX\rangle}{\langle M, M\rangle}\left(\frac{A}{B}\right)_{\langle M\rangle}\right\}\bar{X}_-^{-1}\,{}^t\bar{e}\end{aligned}$$

and by (1.17)

$$\alpha = \left(\frac{A}{B}\right)^2_{\langle M\rangle} e^t X_-^{-1}\frac{\langle dX, M\rangle\langle M, dX\rangle}{\langle M, M\rangle}\bar{X}_-^{-1}\,{}^t\bar{e}.$$

By definition [11, 2.6], $w_1 = \alpha K$, $w_0 = K^2$ and by [11, 2.15]

$$\bar{\partial}(s_1) = w_1 + \frac{1}{2}\left(1 - \frac{A}{B}\right)w_0.$$

Since

$$\frac{C}{B}\psi_1 = \frac{-1}{8\pi^2}\frac{s_1}{\chi\bar{\chi}}, \qquad \chi = (\det H_X)^{-1/2}e_1 e_2,$$

it follows that

$$\bar{\partial}\left(\frac{C}{B}\psi_1\right) = \frac{-1}{8\pi^2}\frac{w_1}{\chi\bar{\chi}} - \frac{1}{2}\left(1 - \left(\frac{A}{B}\right)_{\langle M\rangle}\right)C(E)$$

$$= \left(\frac{-1}{8\pi^2}\right)\left(\frac{A}{B}\right)^3_{\langle M\rangle}\left\{2\left(\frac{A}{B}\right)_{\langle M\rangle}\det\left(\frac{\langle dX, M\rangle\langle M, dX\rangle}{\langle M, M\rangle}\right)\right.$$

$$(2.15)\qquad -\begin{vmatrix} \langle dX_1, dX_1\rangle & \dfrac{\langle dX_1, M\rangle\langle M, dX_2\rangle}{\langle M, M\rangle} \\ \langle dX_2, dX_1\rangle & \dfrac{\langle dX_2, M\rangle\langle M, dX_2\rangle}{\langle M, M\rangle}\end{vmatrix}$$

$$\left. -\begin{vmatrix} \dfrac{\langle dX_1, M\rangle\langle M, dX_1\rangle}{\langle M, M\rangle} & \langle dX_1, dX_2\rangle \\ \dfrac{\langle dX_2, M\rangle\langle M, dX_1\rangle}{\langle M, M\rangle} & \langle dX_2, dX_2\rangle\end{vmatrix}\right\}$$

$$-\frac{1}{2}\left(1 - \left(\frac{A}{B}\right)_{\langle M\rangle}\right)C(E).$$

From (2.13) and (2.15), we have

$$(2.16)\qquad \bar{\partial}\left(\frac{C}{B}\psi_1\right) = \left(\frac{-1}{8\pi^2}\right)\left(\frac{A}{B}\right)^4_{\langle M\rangle}\det\frac{\langle dX, M\rangle\langle M, dX\rangle}{\langle M, M\rangle}$$
$$-\frac{1}{2}\left(\frac{A}{B}\right)^2_{\langle M\rangle}\tilde{C}(E) + \left(\frac{-1}{2} + \left(\frac{A}{B}\right)_{\langle M\rangle}\right)C(E).$$

By (1.21) and (2.16), we have

$$(2.17)\quad \omega(s) = \frac{1}{2}\left\{-2(s+1)\left(\frac{A}{B}\right)^{s+2}_{\langle M\rangle}\tilde{C}(E) + 2(s+2)\left(\frac{A}{B}\right)^{s+1}_{\langle M\rangle}C(E)\right.$$
$$\left. + (s+1)(s+2)\left(\frac{-1}{4\pi^2}\right)\left(\frac{A}{B}\right)^{s+4}_{\langle M\rangle}\det\frac{\langle dX, M\rangle\langle M, dX\rangle}{\langle M, M\rangle}\right\}.$$

By Lemma (2.4), $(B/A)_{\langle M\rangle} = \det(M_{X^\perp}, M_{X^\perp})/(M, M)$, whose value is $((M, M) + (M, X)(X, M))/(M, M)$ on $\tilde{\mathfrak{D}}$, thus $(M, M)(B/A)_{\langle M\rangle}$ for fixed X is a polynomial in M. Now from the expressions of $\tilde{C}(E)$, $C(E)$ and $\omega(s)$ it follows that the form

$$(2.18)\qquad F(M, X) = \left(\frac{B}{A}\right)^{p+3}_{\langle M\rangle}(M, M)^2\omega(p-1)$$

is a polynomial in M. Moreover $F(M, X)$ is defined for every $M \in V$.

In the following we study the evaluation of $F(M, X)$ on $\otimes^2 X^\perp \otimes^2 \overline{X^\perp}$. Let $X \in \tilde{\mathfrak{D}}$, i.e., $(X, X) = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ and $X_V^\perp = \{v \in V \mid (X, v) = 0\}$. Since $(\,,)$ has signature $(p, 2)$, $(\,,)$ induces a positive definite metric on $X_V^\perp$. Let $e_1, \cdots, e_p$ be an orthonormal basis of $X_V^\perp$. As $W = V \oplus V$, $X^\perp$ has a basis

$$(2.19) \qquad \begin{array}{l} (e_1, 0), \cdots, (e_p, 0), \\ (0, e_1), \cdots, (0, e_p). \end{array}$$

Denote by $X_1^\perp$ and $X_2^\perp$ the subspaces of $X^\perp$ spanned by $(e_1, 0), \cdots, (e_p, 0)$ and $(0, e_1), \cdots, (0, e_p)$ respectively. We write $m_i = \langle M, e_i \rangle$, $(i = 1, \cdots, p)$. Then we have $\sum_{i=1}^p m_i \overline{m_i} = \langle M_{X^\perp}, M_{X^\perp} \rangle$ and

$$(2.20) \qquad \left(\frac{B}{A}\right)_{\langle M \rangle} = \sum_{i=1}^{p} m_i \overline{m_i} / (M, M).$$

We give below a table of values of $\tilde{C}(E)$, $C(E)$ and

$$\det \frac{\langle dX, M \rangle \langle M, dX \rangle}{\langle M, M \rangle}$$

on the tensors listed in the left-hand column.

TABLE

	$(-4\pi^2)\tilde{C}(E)$	$(-4\pi^2)C(E)$	$\det \frac{\langle dX, M \rangle \langle M, dX \rangle}{\langle M, M \rangle}$
$\otimes^2 X_1^\perp \otimes^2 \overline{X}^\perp$ $\otimes^2 X_2^\perp \otimes^2 \overline{X}^\perp$ $\otimes^2 X^\perp \otimes^2 \overline{X}_1^\perp$ $\otimes^2 X^\perp \otimes^2 \overline{X}_2^\perp$	0	0	0
$(e_i, 0)(0, e_j)(\bar{e}_k, 0)(0, \bar{e}_l)$ $(e_i, 0)(0, e_j)(0, \bar{e}_k)(\bar{e}_l, 0)$ $(0, e_i)(e_j, 0)(\bar{e}_k, 0)(0, \bar{e}_l)$ $(0, e_i)(e_j, 0)(0, \bar{e}_k)(\bar{e}_l, 0)$ $\{i, j\} \cap \{k, l\} = \varnothing$	0	$\left(\frac{A}{B}\right)^3_{\langle M \rangle} \det \frac{\langle dX, M \rangle \langle M, dX \rangle}{\langle M, M \rangle}$	$\det \frac{\langle dX, M \rangle \langle M, dX \rangle}{\langle M, M \rangle}$
$(e_i, 0)(0, e_i)(\bar{e}_i, 0)(0, \bar{e}_i)$ $(0, e_i)(e_i, 0)(0, \bar{e}_i)(\bar{e}_i, 0)$	2	$2\left(\frac{A}{B}\right)_{\langle M \rangle} - 4\left(\frac{A}{B}\right)_{\langle M \rangle} \frac{\overline{m}_i m_i}{\langle M, M \rangle} + 2\left(\frac{A}{B}\right)^3_{\langle M \rangle} \frac{(\overline{m}_i m_i)^2}{\langle M, M \rangle^2}$	$2\frac{(\overline{m}_i m_i)^2}{\langle M, M \rangle^2}$
$(e_i, 0)(0, e_i)(\bar{e}_i, 0)(0, \bar{e}_j)$, $j \neq i$	0	$-2\left(\frac{A}{B}\right)^2_{\langle M \rangle} \frac{\overline{m}_i m_j}{\langle M, M \rangle} + 2\left(\frac{A}{B}\right)^3_{\langle M \rangle} \frac{\overline{m}_i m_i \overline{m}_i m_j}{\langle M, M \rangle^2}$	$2\frac{\overline{m}_i m_i \overline{m}_i m_j}{\langle M, M \rangle^2}$

TABLE (CONTINUED)

$(e_i,0)(0,e_j)(\bar{e}_i,0)(0,\bar{e}_k)$, i,j,k distinct	0	$-\left(\frac{A}{B}\right)^2_{\langle M\rangle}\frac{\bar{m}_j m_k}{\langle M,M\rangle}+2\left(\frac{A}{B}\right)^3_{\langle M\rangle}\frac{\bar{m}_i m_i \bar{m}_j m_k}{\langle M,M\rangle^2}$	$2\frac{\bar{m}_i m_i \bar{m}_j m_k}{\langle M,M\rangle^2}$
$\sum_{i=1}^{p}(e_i,0)(0,e_j)(\bar{e}_i,0)(0,\bar{e}_k)$, j,k distinct	0	$-p\left(\frac{A}{B}\right)^2_{\langle M\rangle}\frac{\bar{m}_j m_k}{\langle M,M\rangle}$	$2\left(\frac{B}{A}\right)_{\langle M\rangle}\frac{\bar{m}_j m_k}{\langle M,M\rangle}$
$(e_i,0)(0,e_j)(\bar{e}_i,0)(0,\bar{e}_j)$, i,j distinct	1	$\left(\frac{A}{B}\right)_{\langle M\rangle}-\left(\frac{A}{B}\right)^2_{\langle M\rangle}\frac{\bar{m}_i m_i+\bar{m}_j m_j}{\langle M,M\rangle^2}+2\left(\frac{A}{B}\right)^3_{\langle M\rangle}\frac{\bar{m}_i m_i \bar{m}_j m_j}{\langle M,M\rangle^2}$	$2\frac{\bar{m}_i m_i \bar{m}_j m_j}{\langle M,M\rangle^2}$
$\sum_{i=1}^{p}(e_i,0)(0,e_j)(\bar{e}_i,0)(0,\bar{e}_j)$	$p+1$	$p\left(\frac{A}{B}\right)_{\langle M\rangle}-p\left(\frac{A}{B}\right)^2_{\langle M\rangle}\frac{\bar{m}_j m_j}{\langle M,M\rangle}$	$2\left(\frac{B}{A}\right)_{\langle M\rangle}\frac{\bar{m}_j m_j}{\langle M,M\rangle}$

For simplicity, we use the following notation:

$$
(2.21)\quad
\begin{aligned}
a(i,j;\bar{k},\bar{l}) &= (e_i,0)\otimes(0,e_j)\otimes(\bar{e}_k,0)\otimes(0,\bar{e}_l),\\
b(i,j;\bar{k},\bar{l}) &= (e_i,0)\otimes(0,e_j)\otimes(0,\bar{e}_k)\otimes(\bar{e}_l,0),\\
c(i,j;\bar{k},\bar{l}) &= (0,e_i)\otimes(e_j,0)\otimes(\bar{e}_k,0)\otimes(0,\bar{e}_l),\\
d(i,j;\bar{k},\bar{l}) &= (0,e_i)\otimes(e_j,0)\otimes(0,\bar{e}_k)\otimes(\bar{e}_l,0).
\end{aligned}
$$

Here we introduce two subspaces of $\otimes^2 X^\perp \otimes^2 \bar{X}^\perp$. Let $\mathcal{K}$ be the subspace of $\otimes^2 X^\perp \otimes^2 \bar{X}^\perp$ spanned by

$$(2.22)\quad \otimes^2 X_1^\perp \otimes^2 \bar{X}^\perp,\quad \otimes^2 X_2^\perp \otimes^2 \bar{X}^\perp,\quad \otimes^2 X^\perp \otimes^2 \bar{X}_1^\perp,\quad \otimes^2 X^\perp \otimes^2 \bar{X}_2^\perp,$$

$\Sigma_{i=1}^{p} a(i,j;\bar{i},\bar{k})$ $(j\neq k)$, $\Sigma_{i=1}^{p} a(i,j;\bar{i},\bar{j})$ and corresponding elements in terms of b, c, d.

We denote by $\mathfrak{L}$ the subspace of $\otimes^2 X^\perp \otimes^2 \bar{X}^\perp$ spanned by

$$
(2.23)\quad
\begin{aligned}
&a(i,j;\bar{k},\bar{l}) && (\{i,j\}\cap\{k,l\}=\varnothing),\\
&a(j,j;\bar{j},\bar{k})-a(k,j;\bar{k},\bar{k}) && (j\neq k),\\
&a(i,j;\bar{i},\bar{k})-\tfrac{1}{2}a(j,j;\bar{j},\bar{k}) && (i\notin\{j,k\}\text{ and } j\neq k),\\
&a(i,j;\bar{i},\bar{j})-\tfrac{1}{4}\{a(i,i;\bar{i},\bar{i})+a(j,j;\bar{j},\bar{j})\} && (i\neq j)
\end{aligned}
$$

and corresponding elements in terms of b, c, d, which are permutations of the tensors $a(i,j;\bar{k},\bar{l})$.

Proposition (2.6). *We have the following conditions on* $\mathcal{L}$ *and* $\mathcal{K}$.

(i) $\mathcal{K} \oplus \mathcal{L} = \otimes^2 X^\perp \otimes^2 \bar{X}^\perp$,

(ii) $F(M, X) \mid \mathcal{L} = \left(\dfrac{-1}{4\pi^2} \right) \dfrac{(p+1)(p+2)}{2} \det\langle dX, M \rangle \langle M, dX \rangle$,

(iii) $\mathcal{K}$ *is contained in the kernel of* $F(M, X)$.

Proof. (i) Due to our construction, $\mathcal{K} \cap \mathcal{L} = 0$, hence it suffices to show that $\mathcal{K} + \mathcal{L} = \otimes^2 X^\perp \otimes^2 \bar{X}^\perp$. We show that $a(i, j; \bar{k}, \bar{l}) \in \mathcal{K} + \mathcal{L}$. If $\{i, j\} \cap \{k, l\} = \varnothing$, $a(i, j; \bar{k}, \bar{l}) \in \mathcal{L}$. Thus we may assume that $\{i, j\} \cap \{k, l\} \neq \varnothing$. For $j \neq k$,

$$a(j, j; \bar{j}, \bar{k}) \equiv a(k, j; \bar{k}, \bar{k}),$$
$$a(i, j; \bar{i}, \bar{k}) \equiv \tfrac{1}{2} a(j, j; \bar{j}, \bar{k}) \qquad i \notin \{j, k\},$$
$$\sum_{i=1}^{p} a(i, j; \bar{i}, \bar{k}) \equiv 0 \pmod{\mathcal{K} + \mathcal{L}},$$

hence it follows that $a(j, j; \bar{j}, \bar{k}) \in \mathcal{K} + \mathcal{L}$ and consequently $a(i, j; \bar{i}, \bar{k}) \in \mathcal{K} + \mathcal{L}$ for $i = 1, 2, \cdots, p$. We have that

$$(2.24) \qquad \begin{aligned} &a(i, j; \bar{i}, \bar{j}) - \tfrac{1}{4}\{a(i, i; \bar{i}, \bar{j}) + a(j, j; \bar{j}, \bar{j})\} \equiv 0 \qquad (i \neq j), \\ &\sum_{i=1}^{p} a(i, j; \bar{i}, \bar{j}) \equiv 0 \pmod{\mathcal{K} + \mathcal{L}}. \end{aligned}$$

As a consequence

$$\sum_{i=1}^{p} a(i, i; \bar{i}, \bar{i}) + (p+2) a(j, j; \bar{j}, \bar{j}) \equiv 0,$$

$j = 1, 2, \cdots, p$. Since

$$\begin{vmatrix} p+3 & 1 & \cdots & 1 \\ 1 & p+3 & \cdots & 1 \\ & & \ddots & \\ 1 & \cdots & \cdots & p+3 \end{vmatrix} \neq 0,$$

we now conclude by (2.24) that $a(i, i; \bar{j}, \bar{j}) \in \mathcal{K} + \mathcal{L}$ for every i, j. Therefore we have shown that $a(i, j; \bar{k}, \bar{l}) \in \mathcal{K} + \mathcal{L}$. By a similar argument, we also have that $b(i, j; \bar{k}, \bar{l})$, $c(i, j; \bar{k}, \bar{l})$, $d(i, j; \bar{k}, \bar{l}) \in \mathcal{K} + \mathcal{L}$.

(ii) From the table

$$(-4\pi^2)\tilde{C}(E) \,|\, \mathcal{K} + 0,$$

$$(-4\pi^2)C(E) \,|\, \mathcal{K} = \left(\frac{A}{B}\right)^3_{\langle M \rangle} \det \frac{\langle dX, M \rangle \langle M, dX \rangle}{\langle M, M \rangle},$$

and by (2.17) and (2.18), (ii) follows.

(iii) is immediate from the table and (2.23), (2.24).

We summarize the results in the following theorem.

Theorem (2.7). *$F(M, X)$ is a differential form of degree* (2.2) *whose coefficients depend polynomially in M such that if $(M, M) > 0$,*

$$F(M, X) = \left(\frac{B}{A}\right)^{p+3}_{\langle M \rangle} (M, M)^2 \omega_{\langle M \rangle}(p-1);$$

furthermore $\otimes^2 X^\perp \otimes^2 \bar{X}^\perp$ has a splitting $\mathcal{K} + \mathcal{L}$ such that

(i) $F(M, X) \,|\, \mathcal{K} = 0,$

(ii) $F(M, X) \,|\, \mathcal{L} = \left(\frac{-1}{4\pi^2}\right) \frac{(p+1)(p+2)}{2} \det \langle dX, M \rangle \langle M, dX \rangle$

where $\mathcal{K}$, $\mathcal{L}$ are given by (2.22) *and* (2.23).

For applications to §4, it is useful to know that all the *invariant* (2.2) tensors of $X^\perp$ are contained in $\mathcal{K}$. At a point $X \in \tilde{\mathfrak{D}}$ the isotropic subgroup of G is isomorphic to $S(U(p) \times U(2))$, and its action on $X^\perp$ can be expressed in terms of the standard basis (2.19) and matrix product by

$$(2.25) \qquad (g_1 \times g_2)(u_1, u_2) = g_1(u_1, u_2)^t g_2,$$

where $g_1 \times g_2 \in S(U(p) \times U(2))$ and

$$u_1 = \begin{pmatrix} (e_1, 0) \\ \vdots \\ (e_p, 0) \end{pmatrix}, \quad u_2 = \begin{pmatrix} (0, e_1) \\ \vdots \\ (0, e_p) \end{pmatrix}.$$

Lemma (2.8). *The G invariant* (2.2) *tensors of $X^\perp$ are contained in $\mathcal{K}$.*

Proof. We prove that in fact the $S(U(p) \times I)$ invariant tensors of type (2.2) are already in $\mathcal{K}$. By invariant theory (cf. [1, Theorem (3.12)] and [15]), under the standard action of $SU(p)$ on $\mathbf{C}^p$, $\mathrm{Hom}_{SU(p)}(\otimes^r \mathbf{C}^p \otimes^s \bar{\mathbf{C}}^p, \mathbf{C})$ is spanned by the "elementary contractions" and the determinant function. In the present case $S(U(p) \times I)$ acts identically on the two columns of $X^\perp$, and the invariant tensors of type (2.2) are the following, which correspond to the contractions:

$$\left\{ {}^t u_\alpha \otimes \bar{u}_\beta \otimes {}^t u_\delta \otimes \bar{u}_\gamma \right\}, \qquad 1 \leq \alpha, \beta, \delta, \gamma \leq 2.$$

It is easy to check that all these tensors lie in $\mathcal{K}$, e.g.,

$$
\begin{aligned}
{}^t u_1 \otimes \bar{u}_1 \otimes {}^t u_2 \otimes \bar{u}_2 &= \sum_{i,j=1}^{p} a(i, j; \bar{i}, \bar{j}), \\
{}^t u_1 \otimes \bar{u}_2 \otimes {}^t u_2 \otimes \bar{u}_1 &= \sum_{i,j=1}^{p} b(i, j; \bar{i}, \bar{j}), \\
{}^t u_1 \otimes \bar{u}_1 \otimes {}^t u_1 \otimes \bar{u}_2 &\in \otimes^2 X_1^{\perp} \otimes^2 \bar{X}^{\perp}, \quad \text{etc.}
\end{aligned}
$$

This finishes the proof.

For $u, z \in V$, let D_u, $\overline{D_{\bar{u}}}$ be the differential operators

$$D_u = \frac{1}{2\pi i} {}^t u \frac{\partial}{\partial z}, \quad \overline{D_{\bar{u}}} = \frac{1}{2\pi i} {}^t \bar{u} \frac{\partial}{\partial \bar{z}}.$$

We use the notation $e[x] = \exp(2\pi i x)$.

Lemma (2.9). *Let $a, b, c, d \in V$ and t be an indeterminant. Then we have*

$$
\begin{aligned}
& e[-t\langle M, M\rangle] D_a D_b \overline{D_{\bar{c}}}\,\overline{D_{\bar{d}}} e[t\langle M+z, M+z\rangle]\,|_{z=0} \\
&\quad = \frac{t^2}{(-4\pi^2)} \{\langle a, c\rangle\langle b, d\rangle + \langle a, d\rangle\langle b, c\rangle\} \\
&\qquad + \frac{t^3}{(2\pi i)} \{\langle a, d\rangle\langle M, c\rangle\langle b, M\rangle + \langle a, c\rangle\langle M, d\rangle\langle b, M\rangle \\
&\qquad\qquad + \langle b, d\rangle\langle M, c\rangle\langle a, M\rangle + \langle b, c\rangle\langle M, d\rangle\langle a, M\rangle\} \\
&\qquad + t^4 \langle a, M\rangle\langle b, M\rangle\langle M, c\rangle\langle M, d\rangle.
\end{aligned}
$$

Proof. Straightforward.

Now $W = V \oplus V = M_{n2}(\mathbf{C})$ and for $\xi = (a_1, a_2) \otimes (b_1, b_2) \otimes (\bar{c}_1, \bar{c}_2) \otimes (\bar{d}_1, \bar{d}_2) \in \otimes^2 W \otimes^2 \overline{W}$, we define

$$(2.26) \quad D_\xi = D_{a_1} D_{b_2} \overline{D_{\bar{c}_1}}\,\overline{D_{\bar{d}_2}} - D_{a_1} D_{b_2} \overline{D_{\bar{c}_2}}\,\overline{D_{\bar{d}_1}} - D_{a_2} D_{b_1} \overline{D_{\bar{c}_1}}\,\overline{D_{\bar{d}_2}} + D_{a_2} D_{b_1} \overline{D_{\bar{c}_2}}\,\overline{D_{\bar{d}_1}}.$$

Proposition (2.10). *Let $\sum_{i=1}^{l} \lambda_i \xi_i \in \mathcal{L} \subset \otimes^2 W \otimes^2 \overline{W}$; then*

$$
\begin{aligned}
& \sum_{i=1}^{l} \lambda_i D_{\xi_i} e[t(M+z, M+z)]\,|_{z=0} \\
&\quad = \frac{1}{2} t^4 (\det\langle dX, M\rangle\langle M, dX\rangle)\left(\sum_{i=1}^{l} \lambda_i \xi_i\right) e[t(M, M)].
\end{aligned}
$$

Proof. This follows from the definition of $\mathcal{L}$, the fact that $(M+z, M+z) = \langle M+z, M+z\rangle$, and a simple calculation making use of Lemma (2.9).

3. Theta functions and the lifting

In this section we shall use freely the results on the Weil representation and theta functions for the Hermitian pair $U(r, r) \times U(p, q)$ described in [12, §1,3]. For the present paper we restrict to $r = 1$ and $q = 2$. Let k_1 be a totally real algebraic number field with $[k_1 : \mathbf{Q}] = m$, $m > 1$ and k an imaginary quadratic extension of k_1. Let $n = p + 2$, V_1 be an n-dimensional vector space over k, and $(\,,): V_1 \times V_1 \to k$ be a nondegenerate Hermitian form. Set $R = k \otimes_{\mathbf{Q}} \mathbf{R}$, $V = V_1 \otimes_{\mathbf{Q}} \mathbf{R}$ and extend $(\,,)$ to a Hermitian form $(\,,): V \times V \to R$. Clearly $R \simeq \mathbf{C} \oplus \cdots \oplus \mathbf{C}$ (m copies). Let $e_1, \cdots, e_m$ be the irreducible idempotents of R and $V^{(j)} = e_j V$, $1 \leqslant j \leqslant m$. We assume that the signature of $V^{(1)}$ is $(p, 2)$, and those of $V^{(j)}, j > 1$, are $(n, 0)$. Let $G = SU(V, (\,,))$; then

$$G = \prod_{i=1}^{m} SU(V^{(i)}, (\,,)).$$

By our assumption, $\prod_{i=2}^{m} SU(V^{(i)})$ is compact, thus the Hermitian symmetric space associated to G coincides with that of $SU(V^{(1)})$. Let $\tilde{\mathfrak{D}} = \{Z \in V^{(1)} \oplus V^{(1)} \mid (Z, Z) = \left(\begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix}\right)\}$ as in §2. For $Z \in \tilde{\mathfrak{D}}$, we denote

$$Z^{\perp}_{V^{(1)}} = \{X \in V^{(1)} \mid (X, Z) = 0\}. \tag{3.1}$$

Recall that by [12, (1.14)] we have a majorant associated to $(\,,)$ and Z,

$$(X, X)_Z = \begin{cases} (X, X) & \text{if } X \in Z^{\perp}_{V^{(1)}} \oplus \bigoplus_{j=2}^{m} V^{(j)}, \\ -(X, X) & \text{if } X \in \langle Z \rangle, \end{cases} \tag{3.2}$$

where $\langle Z \rangle$ is the R-module spanned by Z_1, Z_2 with

$$Z = (Z_1, Z_2) \in V^{(1)} \oplus V^{(1)}.$$

Now we choose a fixed element i in R with $i^2 = -1$. The Hermitian symmetric space associated to $\mathcal{G}(R) = \{g \in \mathrm{GL}(2, R) \mid {}^t\bar{g}\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)g = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)\}$ is realized as

$$\mathcal{H}(R) = \left\{\tau \in R \mid \frac{1}{i}(\tau - \bar{\tau}) > 0\right\}. \tag{3.3}$$

Here an element $v \in R$ is positive if $e_j v \in \mathbf{C}$ $(j = 1, \cdots, m)$ are all positive. Clearly $\mathcal{H}(R) \simeq \mathcal{H}(\mathbf{C})^m$. On $\mathcal{H}(R)$, $\mathcal{G}$ acts by fractional linear transformation

$$g\tau = (a\tau + b)(c\tau + d)^{-1}$$

for $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$. We have the automorphic factors

$$j(g, \tau) = c\tau + d, \quad j_*(g, \tau) = c\bar{\tau} + d.$$

For $\tau \in \mathcal{H}(R)$ we write

$$\tau = u + iv \tag{3.4}$$

with $u = \bar{u}$, $v = \bar{v} > 0$. As in [12, (1.15)] we define a Schwartz function on V

$$f_{\tau,Z}(M) = e\left[\mathrm{tr}_{R/\mathbf{R}}(u(M, M) + iv(M, M)_Z)\right], \tag{3.5}$$

$\tau \in \mathcal{H}(R)$, $Z \in \tilde{\mathfrak{D}}$, $M \in V$. We denote $M^{(1)} = e_1 M$ which is the component of M in $V^{(1)}$. Now let $M^{(1)} \mapsto F(M^{(1)}, Z)$ be the polynomial in $V^{(1)}$ given by (2.18) with differential forms on $\mathfrak{D}$ of degree (2.2) as values. We have the new Schwartz function

$$\tilde{f}_{\tau,Z}(M) = F(M^{(1)}, Z) f_{\tau,Z}(M). \tag{3.6}$$

Recall the notation: for $x \in R$, and x_i the i-th component in the decomposition $R \simeq \mathbf{C} \oplus \cdots \oplus \mathbf{C}$ (m copies) and $t = (t_1, \cdots, t_m) \in \mathbf{Z}^m$, then x^t stands for $x_1^{t_1} \cdots x_m^{t_m}$. For $g \in \mathcal{G}(R)$ we also recall that $\varepsilon(g)$ is given by [12, Proposition (1.4)] with $r = 1$. Let $K(R)$ be the isotropic subgroup of $\mathcal{G}(R)$ at i, and χ be a character of $K(R)$ given by

$$\chi(g) = j(g, i).$$

Let Ω^0, Ω', Ω be the sets defined in [12, §1], and let $r(g)$, $r_0(g)$ be the unitary operators defined in [12, (1.10)] and [12, following Proposition (1.4)].

Proposition (3.1). *For $g \in \Omega^0 \cup \Omega' \cup \Omega$,*

$$r(g)\tilde{f}_{\tau,Z}(M) = \varepsilon(g) j(g, \tau)^{-P} j_*(g, \tau)^{-Q} \tilde{f}_{g\tau,Z}(M),$$

where $P = (p + 4, p + 2, \cdots, p + 2)$ and $Q = (2, 0, \cdots, 0)$. In particular for $k \in K(R)$,

$$r_0(k)\tilde{f}_{i,Z}(M) = \chi^{-(p+2)}(k)\tilde{f}_{i,Z}(M),$$

where $\chi^{-(p+2)}$ stands for $\chi^{-(p+2, \cdots, p+2)}$.

Proof. Let $Z \in \tilde{\mathfrak{D}}$ and let $\pi(Z)$ be its image in $\mathfrak{D}$. By Lemma (2.1), $Z^{\perp}$ is identified with the tangent space of $\mathfrak{D}$ at $\pi(Z)$. By Theorem (2.7), $\otimes^2 Z^{\perp} \otimes^2 \bar{Z}^{\perp}$ has a decomposition $\mathcal{H} \oplus \mathcal{L}$ on which $F(M^{(1)}, Z)$ satisfies the following conditions:

(i) The restriction of $F(M^{(1)}, Z)$ to $\mathcal{H}$ vanishes identically.

(ii) For its restriction to $\mathcal{L}$, we have

$$F(M^{(1)}, Z)\,|_{\mathcal{L}} = \left(\frac{-1}{4\pi^2}\right) \frac{(p+1)(p+2)}{2} \det\langle dZ, M^{(1)}\rangle \langle M^{(1)}, dZ\rangle.$$

Therefore it suffices to show the proposition for $\tilde{f}_{\tau,Z}(M)\,|\,\mathcal{L}$. For $\xi = \sum \lambda_i \xi_i \in \mathcal{L} \subset \otimes^2(V^{(1)} \oplus V^{(1)}) \otimes^2 \overline{(V^{(1)} \oplus V^{(1)})}$ we define the 4-th order differential operators $D_\xi = \sum_i \lambda_i D_{\xi_i}$ by (2.26). By Proposition (2.10), we have

$$D_\xi\{e[\mathrm{tr}_{R/\mathbf{R}}(M+Z, M+Z)c^{-1}d]\, f_{\tau,Z}(M+Z)\}_{Z=0}$$

$$(3.7) = \frac{(-4\pi^2)}{(p+1)(p+2)}\left(\frac{c\tau+d}{c}\right)^{(4,0,\cdots,0)} e[\mathrm{tr}_{R/\mathbf{C}}(M, M)c^{-1}d]\, \tilde{f}_{\tau,Z}(M)(\xi).$$

Then by [12, Proposition (1.4)] and a standard argument as in [5, Proposition (4.2)] and [8, Lemma (8.3)], we have the relation

$$r(g)\tilde{f}_{\tau,Z}(M)(\xi)$$
$$= \varepsilon(g)j(g,\tau)^{-(p+4,p+2,\cdots,p+2)} j_*(g,\tau)^{-(2,0,\cdots,0)} \tilde{f}_{g\tau,Z}(M)(\xi), \qquad \xi \in \mathfrak{L}.$$

Let $\mathcal{O}$ be the ring of integers of k, L an $\mathcal{O}$-lattice of V_1, and L^* its dual lattice given by $L^* = \{v \in V \mid \mathrm{tr}_{R/\mathbf{R}}(v, L) \subset \mathbf{Z}\}$. Now as in [12], we can define our basic theta function. For $\tau = u + iv \in \mathcal{H}(R)$, let

$$\sigma_\tau = \begin{pmatrix} v^{1/2} & uv^{-1/2} \\ 0 & v^{-1/2} \end{pmatrix} \in \mathcal{G}(R),$$

so that $\sigma_\tau(i) = \tau$. By Proposition (3.1),

$$r(k)\tilde{f}_{i,Z} = \chi^{-(p+2)}(k)\tilde{f}_{i,Z}.$$

We construct a theta function for $h \in L^*/L$ by

$$(3.8)\qquad \begin{aligned}\Theta(\tau,h,Z) &= \sum_{M\equiv h(L)} N_{R/\mathbf{C}}(v)^{-((p+2)/2)} r(\sigma_\tau)\tilde{f}_{i,Z}(M)\\ &= v^{(2,0,\cdots,0)} \sum_{M\equiv h(L)} \tilde{f}_{\tau,Z}(M).\end{aligned}$$

By [12, Proposition (1.6)], there exists a positive integer N such that

$$(3.9)\qquad \Theta(\gamma\tau,h,Z) = j(\gamma,\tau)^{p+2}\Theta(\tau,h,Z),$$

for $\gamma \in \tilde{\Gamma}(N) \subset \mathcal{G}(\mathcal{O})$, where $p+2$ stands for $(p+2,\cdots,p+2)$.

Let $\tau = u + iv \in \mathcal{H}(R)$ and $g \in \mathcal{G}(R)$. From

$$g\tau - \overline{g\tau} = j\overline{(g,\tau)}^{-1}(\tau - \bar{\tau})j(g,\tau)^{-1}$$

one deduces $\mathrm{Im}(g\tau) = \mathrm{Im}(\tau)(j(g\tau)j\overline{(g,\tau)})^{-1}$. The invariant measure on $\mathcal{H}(R)$ is given by

$$(3.10)\qquad N_{R/\mathbf{R}}(v)^{-1}\{dv\}\{dv\}.$$

Lemma (3.2). *Let $\eta \in R$ with $\bar{\eta} = \eta > 0$. Then*

$$\int e^{-2\pi \mathrm{tr}_{R/\mathbf{C}}(\eta v)} v^{(l_1,\cdots,l_m)}\{dv\} = \prod_{i=1}^{m} \Gamma_1(l_i+1)\eta^{-(l_1+1,\cdots,l_m+1)},$$

where $\Gamma_1(l_j+1) = (2\pi)^{-(l_j+1)}\Gamma(l_j+1)$.

Proof. This is a special case of [12, Lemma (3.2)].

Now let

$$\tilde{\Gamma} = \tilde{\Gamma}(N),$$
$$S_1(k) = \{a \in k \mid \bar{a} = a\},$$
$$S_1(N\mathcal{O}) = \{a \in N\mathcal{O} \mid \bar{a} = a\},$$
$$(3.11) \qquad S_1^*(N\mathcal{O}) = \{\eta \in S_1(k) \mid \operatorname{tr}_{R/\mathbf{C}}(\eta S_1(N\mathcal{O})) \subset \mathbf{Z}\},$$
$$C_N = \operatorname{vol}(S_1(R)/S_1(N\mathcal{O})),$$
$$\tilde{\Gamma}_\infty = \tilde{\Gamma} \cap \left\{ \begin{pmatrix} \pm 1 & \\ 0 & \pm 1 \end{pmatrix} \middle| \beta \in S_1(N\mathcal{O}) \right\}.$$

For $\tau \in \mathcal{H}(R)$, $s \in \mathbf{C}$ and $\eta \in S_1^*(N\mathcal{O})$ with $\eta > 0$ define

$$(3.12) \quad \phi_{\eta,s}(\tau) = C_N^{-1} \sum_{\tilde{\Gamma}_\infty \setminus \tilde{\Gamma}} j(\gamma,\tau)^{-(p+2)} \left(N_{R/\mathbf{C}} \operatorname{Im}(\gamma\tau)^2\right) \cdot e\left[\operatorname{tr}_{R/\mathbf{C}}(\eta\gamma\tau)\right],$$

where $p + 2 = (p + 2, \cdots, p + 2)$. As in [12, §3] this series is absolutely convergent for $2re(s) + p + 2 > 2$, and $\phi_{\eta,0}$ is the holomorphic Poincaré series associated to η. The functions $\phi_{\eta,0}$, $\eta > 0$, $\eta \in S_1^*(N\mathcal{O})$ span the space $\mathbb{S}_{p+2}(\tilde{\Gamma})$ of Hermitian (Hilbert) cusp forms of weight $(p + 2, \cdots, p + 2)$ for $\tilde{\Gamma}$. The Petersson product on $\mathbb{S}_{p+2}(\tilde{\Gamma})$ is given by

$$(3.13) \quad \langle \phi_1, \phi_2 \rangle = \int_{\tilde{\Gamma} \setminus \mathcal{H}(R)} \overline{\phi_1(\tau)}\, \phi_2(\tau) v^{(p+2,\cdots,p+2)} N_{R/\mathbf{C}}(v)^{-2} \{du\}\{dv\},$$

for $\phi_1, \phi_2 \in \mathbb{S}_{p+2}(\tilde{\Gamma})$. The Fourier expansion for $\phi \in \mathbb{S}_{p+2}(\tilde{\Gamma})$ is

$$(3.14) \qquad \phi(\tau) = \sum_{\substack{\eta \in S_1^*(N\mathcal{O}) \\ \eta > 0}} a(\eta) e\left[\operatorname{tr}_{R/\mathbf{C}}(\eta\tau)\right],$$

and

$$(3.15) \quad \langle \phi_{\eta,\bar{s}}, \phi \rangle = 2^{-m(p+1+s)} \Gamma_1(p + 1 + s)^m \eta^{-(p+1+s,\cdots,p+1+s)} a(\eta).$$

Finally, by (3.2), for $M \in V$

$$(3.16) \quad (M, M) + (M, M)_Z = 2\left(M_{Z^\perp}^{(1)}, M_{Z^\perp}^{(1)}\right) + 2 \sum_{i=2}^{m} (M^{(i)}, M^{(i)}).$$

Theorem (3.3). *Let* $\Theta = \Theta(\tau, h, Z)$ *be as in* (3.8). *Then for* $p + 2 = (p + 2, \cdots, p + 2)$ *and* $\operatorname{Re}(s) > 1$,

$$\langle \phi_{\eta,\bar{s}}, \Theta(\tau, h, Z) \rangle = 2^{-(m-1)(s+p+1)} 2^{-(s+p+3)} \Gamma_1(s + p + 1)^{m-1} \Gamma_1(s + p + 3)$$
$$\cdot N_{R/\mathbf{C}}(\eta)^{-(s+p+1)} \sum_{\substack{x \equiv h(L) \\ (M,M) = \eta}} \left(\frac{A}{B}\right)_{\langle M \rangle}^{s} \omega_{\langle M \rangle}(p - 1).$$

Proof.

$$\langle \phi_{\eta,\bar{s}}, \Theta \rangle = \int_{\tilde{\Gamma} \setminus \mathcal{H}(R)} \Theta(\tau, h, Z) \bar{\phi}_{\eta,\bar{s}}(\tau) v^{(p,\cdots,p)} \{du\} \{dv\}$$

$$= C_N^{-1} \int_{\tilde{\Gamma}_\infty \setminus \mathcal{H}(R)} \Theta(\tau, h, Z) \exp\left(-2\pi i \operatorname{tr}_{R/\mathbf{C}} \eta u - 2\pi \operatorname{tr}_{R/\mathbf{C}}(\eta v)\right) \cdot v^{(s+p,\cdots,s+p)} \{du\} \{dv\}.$$

Let the fundamental domain of $\tilde{\Gamma}_\infty$ be

$$\{\tau = u + iv \in \mathcal{H}(R) \mid u \in S_1(R)/S_1(N\mathcal{O})\}.$$

The integral becomes

$$C_N^{-1} \sum_{M \equiv h(L)} F(M^{(1)}, Z) \int_{v>0} \int_{S_1(R)/S_1(N\mathcal{O})} \exp(2\pi i (u(M, M) - \eta)) \cdot \{du\} \exp\left(-2\pi \operatorname{tr}_{R/\mathbf{C}}(\eta v + (M, M)_Z v)\right) v^{(2,0,\cdots,0)} v^{(s+p,\cdots,s+p)} \{dv\}$$

$$= \sum_{\substack{M \equiv h(L) \\ (M,M)=\eta}} F(M^{(1)}, Z) \int_{v>0} \exp\left(-2\pi \operatorname{tr}_{R/\mathbf{C}}((M, M) + (M, M)_Z) v\right) \cdot v^{(s+p+2, s+p, \cdots, s+p)} \{dv\}$$

$$= \left(\sum_{M \equiv h(L)} F(M^{(1)}, Z) \left(\frac{(M^{(1)}, M^{(1)})}{(M_Z^{(1)}, M_Z^{(1)})} \right)^{(s+p+3)} N_{R/\mathbf{C}}((M, M))^{-(s+p+1)} \cdot (M^{(1)}, M^{(1)})^{-2} \right)$$

$$\cdot 2^{-(m-1)(s+p+1)} 2^{-(s+p+3)} \Gamma_1(s + p + 1)^{m-1} \Gamma_1(s + p + 3)$$

$$= 2^{-(m-1)(s+p+1)} 2^{-(s+p+3)} \Gamma_1(s + p + 1)^{m-1} \Gamma_1(s + p + 3) \cdot N_{R/\mathbf{C}}(\eta)^{-(s+p+1)} \sum_{\substack{X \equiv h(L) \\ (M,M)=\eta}} \left(\frac{A}{B} \right)^s_{\langle M \rangle} \omega_{\langle M \rangle}(p - 1)$$

$$= (4\pi)^{-m(s+p+1)-2} \Gamma(s + p + 1)^{m-1} \Gamma(s + p + 3) N_{R/\mathbf{C}}(\eta)^{-(s+p+1)} \sum_{\substack{M \equiv h(L) \\ (M,M)=\eta}} \left(\frac{A}{B} \right)^s_{\langle M \rangle} \omega_{\langle M \rangle}(p - 1).$$

As in [12] we need $\operatorname{re}(s) > 1$ to guarantee the interchangeability of integral and summation signs.

Recall $G = SU(V, (\,,\,)) = \prod_{i=1}^{m} SU(V^{(i)})$. Let $\Gamma = \{\gamma \in G \mid \gamma L = L$ and γ acts trivially on $L^*/L\}$. Replacing Γ by a subgroup of finite index if necessary, we may assume that Γ is torsion free. Since $\prod_{i=2}^{m} SU(V^{(i)})$ is compact, we identify Γ with its image in $SU(V^{(1)})$. By our hypothesis $SU(V^{(1)})/\Gamma$ is compact, or equivalently $\Gamma\backslash\mathfrak{D}$ is compact (this can be achieved by assuming that $(V_1, (\,,\,))$ is anisotropic). We are thus in a position to make use of the dual form $\hat{\omega}_{\langle M\rangle}(s)$ constructed in §1. For $h \in L^*$, let

$$L_{\eta,h} = \{M \in L^* \mid M \equiv h(L), (M, M) = \eta\}.$$

We know that $L_{\eta,h}$ is Γ-invariant and has only finitely many Γ-orbits,

$$L_{\eta,h} = \bigcup_{i=1}^{l} \Gamma M_i. \tag{3.17}$$

We denote

$$\hat{\omega}_\eta(s) = \sum_{M \in L_{\eta,h}} \omega_{\langle M^{(1)}\rangle}(s). \tag{3.18}$$

Then it follows readily that

$$\begin{aligned}
\hat{\omega}_\eta(s) &= \sum_{i=1}^{l} \sum_{M \in \Gamma M_i} \omega_{\langle M^{(1)}\rangle}(s) \\
&= \sum_{i=1}^{l} \sum_{\Gamma_{M_i}\backslash\Gamma} \gamma^* \omega_{\langle M_i^{(1)}\rangle}(s) \\
&= \sum_{i=1}^{l} \left[\Gamma_{\langle M_i\rangle} : \Gamma_{M_i}\right] \sum_{\Gamma_{\langle M_i^{(1)}\rangle}\backslash\Gamma} \gamma^* \omega_{\langle M_i^{(1)}\rangle}(s) \\
&= \sum_{i=1}^{l} \left[\Gamma_{\langle M_i\rangle} : \Gamma_{M_i}\right] \hat{\omega}_{\langle M_i^{(1)}\rangle}(s).
\end{aligned}$$

By §1, $\hat{\omega}_{\langle M_i^{(1)}\rangle}(s)$ converges for $\mathrm{Re}(s) > p - 1$ and is dual to the cycle $C_{\langle M_i^{(1)}\rangle}$ which is the image of $\mathfrak{D}_{\langle M_i^{(1)}\rangle}$ in $\Gamma\backslash\mathfrak{D}$. Thus $\hat{\omega}_\eta(s)$ is dual to

$$C_\eta = \sum_{i=1}^{l} \left[\Gamma_{\langle M_i\rangle} : \Gamma_{M_i}\right] C_{\langle M_i^{(1)}\rangle}. \tag{3.19}$$

Similarly the result in Theorem (3.3) can be expressed as

$$\begin{aligned}
\left\langle \phi_{\eta,\bar{s}}, \Theta(\tau, h, Z)\right\rangle &= (4\pi)^{-m(s+p+1)-2}\Gamma(s+p+1)^{m-1} \\
&\cdot \Gamma(s+p+3) N_{R/\mathbf{C}}(\eta)^{-(s+p+1)} \\
&\cdot \sum_{i=1}^{l} \left[\Gamma_{\langle M_i\rangle} : \Gamma_{M_i}\right] \sum_{\Gamma_{\langle M_i^{(1)}\rangle}\backslash\Gamma} \gamma^* \left(\left(\frac{A}{B}\right)^{s}_{\langle M_i^{(1)}\rangle} \omega_{\langle M_i^{(1)}\rangle}(p-1)\right).
\end{aligned} \tag{3.20}$$

Note that since the series (3.18) defining $\hat{\omega}_\eta(s)$ does not converge at $s = p - 1$, one must compare its analytic continuation at that point with the continuation of $\langle \phi_{\eta,\bar{s}}, \Theta(\tau, h, Z)\rangle$ at $s = 0$. This is carried out in the next section.

4. Cohomological interpretation of lifting

From (3.20), to give a geometric interpretation to the lifting we need to study $\Sigma_{\gamma \in \Gamma_1 \setminus \Gamma} \gamma^*((A/B)^s \omega(p-1))$ at $s = 0$. We denote

$$f(s) = \sum_{\gamma \in \Gamma_1 \setminus \Gamma} \gamma^* \left(\left(\frac{A}{B} \right)^{s-p+1} \omega(p-1) \right). \tag{4.1}$$

From (1.30) and (1.31)

$$\begin{aligned} \bar{\partial}\left(\frac{C}{B} \psi_1 \right) &= -\frac{1}{2} \left(\frac{A}{B} \right)^2 \xi_1 - \frac{1}{2} \left(1 - \frac{A}{B} \right) C(E), \\ \tilde{C}(E) &= \frac{B}{A} C(E) + \xi_1 + 2\xi_2, \end{aligned} \tag{4.2}$$

where at $Z_1 = 0$

$$\begin{aligned} (-4\pi^2)\xi_1 &= \left(\frac{B}{A} \right)^2 \big[{}^t dw_1 \wedge d\bar{w}_1 \wedge dv_2 \wedge d\bar{v}_2 + {}^t dw_2 \wedge d\bar{w}_2 \wedge dv_1 \wedge d\bar{v}_1 \\ &\qquad - {}^t dw_2 \wedge d\bar{w}_1 \wedge dv_1 \wedge d\bar{v}_2 - {}^t dw_1 \wedge d\bar{w}_2 \wedge dv_2 \wedge d\bar{v}_1 \big], \\ (-4\pi^2)\xi_2 &= \left(\frac{B}{A} \right)^3 dv_1 \wedge d\bar{v}_1 \wedge dv_2 \wedge d\bar{v}_2. \end{aligned} \tag{4.3}$$

Substituting (4.3) in (1.21) we have

Lemma (4.1).

$$\begin{aligned} \omega(s) &= \frac{1}{2} \left(\frac{A}{B} \right)^{s+2} \Big\{ s(s+1)\tilde{C}(E) + (1-s)(s+2)\frac{B}{A} C(E) \\ &\qquad - (s+1)(s+2)\xi_1 \Big\} \\ &= \left(\frac{A}{B} \right)^{s+2} \Big\{ -(s+1)\tilde{C}(E) + (s+2)\frac{B}{A} C(E) + (s+1)(s+2)\xi_2 \Big\}. \end{aligned}$$

For a G_1 invariant differential form h with bounded norm, we define

$$h_s = \sum_{\Gamma_1 \setminus \Gamma} \gamma^* \left(\left(\frac{A}{B} \right)^{s+2} h \right). \tag{4.4}$$

The series is absolutely convergent for $\mathrm{Re}(s) > p-1$ and has at most a simple pole at $s - p - 1$ since this is so for the series $\Sigma_{\Gamma_1 \setminus \Gamma}\, \gamma^*(A/B)^{s+2}$. We write

(4.5) $$\mathrm{res}(h) = \text{residue of } h_s \text{ at } s = p-1.$$

Lemma (4.2). *The residue of* $\Sigma_{\Gamma_1 \setminus \Gamma}\, \gamma^*(A/B)^{s+2}$ *at* $s = p-1$ *is*

$$\pi^2 \operatorname{vol}(\Gamma_1 \setminus \mathfrak{D}_1)/\operatorname{vol}(\Gamma \setminus \mathfrak{D});$$

in particular it is a constant function.

Proof. From results of §1,

$$\Lambda^2\omega(s) = (-i\Lambda)\left\{\left(\frac{-1}{4\pi^2}\right)(s-p+1)[-(a+b)+(s+1)(c+d)]\right\}\left(\frac{A}{B}\right)^{s+2}$$

$$= \frac{(s-p+1)(s-p+2)}{2\pi^2}\left(\frac{A}{B}\right)^{s+2}.$$

It follows that

$$\mathrm{res}\left(\sum_{\Gamma_1 \setminus \Gamma} \gamma^*\left(\frac{A}{B}\right)^{s+2}\right) = 2\pi^2\Lambda^2\hat{\omega}(p-1),$$

and it remains to prove

$$\Lambda^2\hat{\omega}(p-1) = \frac{\operatorname{vol}(\Gamma_1 \setminus \mathfrak{D}_1)}{2\operatorname{vol}(\Gamma \setminus \mathfrak{D})}.$$

This equality follows from (cf. [12, proof of Corollary (4.4)])

$$\Lambda^2\hat{\omega}(p-1) = \frac{2}{2^{2p}(2p-2)!\operatorname{vol}(\Gamma \setminus \mathfrak{D})}\int_{\Gamma_1 \setminus \mathfrak{D}_1} \kappa^{2p-2},$$

and the fact [4, p. 90]

$$\kappa^{2p-2}\big|_{\mathfrak{D}_1} = 2^{2p-2}(2p-2)!\,dv_{\mathfrak{D}_1}.$$

Corollary (4.3). *For a G invariant form* ν,

$$\mathrm{res}(\nu) = \sigma\nu, \quad \textit{where } \sigma = \pi^2\operatorname{vol}(\Gamma_1 \setminus \mathfrak{D}_1)/\operatorname{vol}(\Gamma \setminus \mathfrak{D}).$$

Lemma (4.4). *The form* $f(s)$ *is regular at* $s = p-1$ *and moreover*

$$f(p-1) = \hat{\omega}(p-1) + \sigma\tilde{C}(E) - \mathrm{res}\left(\frac{B}{A}C(E)\right) - (2p+1)\mathrm{res}(\xi_2)$$

$$= \hat{\omega}(p-1) - \frac{\sigma(2p-1)}{2}\tilde{C}(E) + \frac{(2p-1)}{2}\mathrm{res}\left(\frac{B}{A}C(E)\right)$$

$$+\frac{1}{2}(2p+1)\mathrm{res}(\xi_1).$$

Proof. We have that

$$\left(\frac{A}{B}\right)^{s-p+1}\omega(p-1)$$
$$=\frac{1}{2}\left(\frac{A}{B}\right)^{s+2}\left\{(p-1)p\tilde{C}(E)-(p-2)(p+1)\frac{B}{A}C(E)-p(p+1)\xi_1\right\}$$
$$=\left(\frac{A}{B}\right)^{s+2}\left\{-p\tilde{C}(E)+(p+1)\frac{B}{A}C(E)+p(p+1)\xi_2\right\}.$$

It follows that

$$\begin{aligned}\left(\frac{A}{B}\right)^{s-p+1}\omega(p-1)&=\omega(s)-\frac{1}{2}(s+1-p)(s+p)\left(\frac{A}{B}\right)^{s+2}\tilde{C}(E)\\&\quad+\frac{1}{2}(s+1-p)(s+p)\left(\frac{A}{B}\right)^{s+2}\left(\frac{B}{A}C(E)\right)\\&\quad+\frac{1}{2}(s+1-p)(s+p+2)\left(\frac{A}{B}\right)^{s+2}\xi_1\\&=\omega(s)+\ (s+1-p)\left(\frac{A}{B}\right)^{s+2}\tilde{C}(E)\\&\quad-(s+1-p)\left(\frac{A}{B}\right)^{s+2}\left(\frac{B}{A}C(E)\right)\\&\quad-(s+1-p)(s+p+2)\left(\frac{A}{B}\right)^{s+2}\xi_2,\end{aligned}$$

and the assertions for $f(p-1)$ are immediate.

In the following, we study $\operatorname{res}(BC(E)/A)$, $\operatorname{res}(\xi_1)$ and $\operatorname{res}(\xi_2)$. Their cohomological meaning will give us the desired interpretation. Since $\hat{\omega}(s)$ is regular at $s=p-1$, we derive easily from Lemma (4.1) the following lemma on residues.

Lemma (4.5).

$$\text{(i)}\quad \left(\frac{p-1}{p+1}\right)\sigma\tilde{C}(E)+\frac{2-p}{p}\operatorname{res}\left(\frac{B}{A}C(E)\right)-\operatorname{res}(\xi_1)=0,$$

$$\text{(ii)}\quad \left(\frac{-1}{p+1}\right)\sigma\tilde{C}(E)+\frac{1}{p}\operatorname{res}\left(\frac{B}{A}C(E)\right)+\operatorname{res}(\xi_2)=0.$$

The following lemma is an easy consequence of Kuga's formula of the Laplacian operator on symmetric spaces.

Lemma (4.6). *Let $\mathfrak{D}$ be a symmetric space, and G the identity component of the group of isometries of $\mathfrak{D}$. Let f and h be differential forms of $\mathfrak{D}$ such that f is harmonic and h is G invariant; then $f\wedge h$ is harmonic.*

We now let a, b, c, d denote the G_1 invariant forms defined in (1.27). For forms f and h we denote

$$\varepsilon(f, h) = (i\Lambda)(f \wedge h) - (i\Lambda f) \wedge h - f \wedge (i\Lambda h). \tag{4.6}$$

We also recall from Lemmas (1.4) and (1.5) that

$$(i\Lambda)(a+b) = 2(p-1), \quad (i\Lambda)(c+d) = 2, \tag{4.7}$$
$$(i\Lambda)\tilde{C}(E) = \left(\frac{-1}{4\pi^2}\right)(p+1)(-\partial\bar{\partial}\log A).$$

We list below a table of values of $\varepsilon(f, h)$ which are proved in the same way as Lemmas (1.4) and (1.5).

f	h	$\varepsilon(f, h)$
$a+b$	$\frac{B}{A}C(E)$	$-4(\frac{B}{A})C(E)$
$c+d$	$\frac{B}{A}C(E)$	0
$a+b$	ξ_1	$-2\xi_1$
$c+d$	ξ_1	$-2\xi_1$
$a+b$	ξ_2	0
$c+d$	ξ_2	$-2\xi_2 - \frac{(c+d)^2}{(-4\pi^2)}$

Lemma (4.7). *The form* $(i\Lambda)(\hat{\omega}(p-1))$ *is harmonic and*

$$(i\Lambda)(\hat{\omega}(p-1)) = \left(\frac{-1}{4\pi^2}\right)\text{res}\{-(a+b) + p(c+d)\}.$$

Proof. Since $\hat{\omega}(p-1)$ is harmonic, $(i\Lambda)\hat{\omega}(p-1)$ is also harmonic. The second assertion follows from (1.33).

The form $\tilde{C}(E)$ is G invariant and by Lemma (4.6), $((i\Lambda)\hat{\omega}(p-1))\tilde{C}(E)$ is harmonic. In the following, we compute $(i\Lambda)$ $\{((i\Lambda)\hat{\omega}(p-1))\tilde{C}(E)\}$. By the preceding lemma, we have

$$((i\Lambda)\hat{\omega}(p-1))\tilde{C}(E) = \left(\frac{-1}{4\pi^2}\right)\text{res}\{[-(a+b) + p(c+d)]\tilde{C}(E)\}, \tag{4.8}$$

for $\tilde{C}(E)$ is G invariant. Thus

$$(i\Lambda)\{[(i\Lambda)\hat{\omega}(p-1)]\tilde{C}(E)\}$$
$$= \left(\frac{-1}{4\pi^2}\right)\text{res}\{(i\Lambda)[(-(a+b) + p(c+d))\tilde{C}(E)]\},$$

and from the table the above value is

$$\left(\frac{-1}{4\pi^2}\right)\operatorname{res}\Big\{2\tilde{C}(E)+\left(\frac{-1}{4\pi^2}\right)(p+1)[-(a+b)+p(c+d)]$$
$$\cdot(-\partial\bar{\partial}\log A)+4C(E)\frac{B}{A}+2(1-p)\xi_1-p\left[4\xi_2+\frac{4cd}{(-4\pi^2)}+\frac{2c^2}{(-4\pi^2)}\right]\Big\}$$

$$(4.9)\qquad =\left(\frac{-1}{4\pi^2}\right)2\sigma\tilde{C}(E)+\left(\frac{-1}{4\pi^2}\right)(p+1)((i\Lambda)\hat{\omega}(p-1))(-\partial\bar{\partial}\log A)$$
$$+\left(\frac{-1}{4\pi^2}\right)4\operatorname{res}\left(\frac{B}{A}C(E)\right)+\left(\frac{-1}{4\pi^2}\right)2(1-p)\operatorname{res}(\xi_1)$$
$$-8p\left(\frac{-1}{4\pi^2}\right)\operatorname{res}(\xi_2).$$

Lemma (4.8).

$$4\operatorname{res}\left(\frac{B}{A}C(E)\right)+2(1-p)\operatorname{res}(\xi_1)-8p\operatorname{res}(\xi_2)$$
$$=(-4\pi)^2(i\Lambda)\{[i\Lambda\hat{\omega}(p-1)]\tilde{C}(E)\}$$
$$-2\sigma\tilde{C}(E)-(p-1)[i\Lambda\hat{\omega}(p-1)](-\partial\bar{\partial}\log A).$$

From Lemmas (4.5) and (4.8) we have a system of three linear equations in the three unknowns $\operatorname{res}(\xi_1)$, $\operatorname{res}(\xi_2)$ and $\operatorname{res}(BC(E)/A)$. Solving this system we obtain

$$\operatorname{res}\left(\frac{B}{A}C(E)\right)+(2p+1)\operatorname{res}(\xi_2)=\left[1+\frac{p}{(p+1)(p+2)}\right]\sigma\tilde{C}(E)$$
$$(4.10)\qquad -\frac{(-4\pi^2)}{2(p+2)}(i\Lambda)\{[i\Lambda\hat{\omega}(p-1)]\tilde{C}(E)\}$$
$$-\frac{(p+1)}{2(p+2)}[i\Lambda\hat{\omega}(p-1)](-\partial\bar{\partial}\log A).$$

We now summarize all the known results concerning $f(p-1)$.

Theorem (4.9). *The family of differential forms* $\Sigma_{\gamma\in\Gamma_1\setminus\Gamma}\gamma^*((A/B)^s\omega(p-1))$ *is regular at* $s=0$ *and its value there is the harmonic form*

$$\hat{\omega}(p-1)-\frac{2\pi^2}{p+2}[\Lambda\hat{\omega}(p-1)][\Lambda\tilde{C}(E)]$$
$$(4.11)\qquad +\frac{2\pi^2}{p+2}\Lambda\{[\Lambda\hat{\omega}(p-1)]\tilde{C}(E)\}-\frac{2p\pi^2}{(p+1)(p+2)}[\Lambda^2\hat{\omega}(p-1)]\tilde{C}(E).$$

This harmonic form is primitive, and is orthogonal in the Hodge inner product to all G invariant forms

Proof. The regularity at $s = 0$ follows from Lemma (4.4). Also by that lemma and (4.10) and (4.7), $f(p-1)$ has the value (4.11) if one notes that in the proof of Lemma (4.2),

$$\Lambda^2\hat{\omega}(p-1) = \frac{\operatorname{vol}(\Gamma_1 \backslash \mathfrak{D}_1)}{2\operatorname{vol}(\Gamma \backslash \mathfrak{D})}. \tag{4.12}$$

To prove the last statement of the theorem, we first show $\Lambda f(p-1) = 0$. For this we use the first formula in Lemma (4.4) and Lemmas (1.5)(i)(ii), (1.7)(iv), and (4.7) to derive

$$\begin{aligned} i\Lambda f(p-1) &= \frac{-1}{4\pi^2}\operatorname{res}\{-(a+b) + p(c+d)\} \\ &\quad + \sigma\left(\frac{-1}{4\pi^2}\right)(p+1)(-\partial\bar{\partial}\log A) - \operatorname{res}\left\{\left(\frac{-1}{4\pi^2}\right)p(a+b)\right\} \\ &\quad -(2p+1)\left(\frac{-1}{4\pi^2}\right)\operatorname{res}(c+d) = 0. \end{aligned}$$

Finally let

$$\rho\colon T^*(\mathfrak{D}) \to \bar{T}(\mathfrak{D})$$

be the antilinear isomorphism defined by the G invariant metric. Then ρ takes invariant forms to invariant tensors, and by [4, pp. 92–93]

$$f(p-1)\Lambda * \bar{h} = f(p-1)(\rho(h))2^{2p}dV_{\mathfrak{D}}. \tag{4.13}$$

To prove the orthogonality of $f(p-1)$ to invariant forms, it now suffices by (4.1) to show that $\omega(p-1)$ vanishes on invariant tensors, but that is a consequence of Theorem (2.7) and Lemma (2.8).

This theorem shows that it is appropriate to introduce a normalized G invariant form and a linear operator on $\mathcal{H}^{2,2}(\Gamma \backslash \mathfrak{D})$, the space of harmonic forms of degree (2, 2), given by (cf. (4.6))

(4.14)

$$\text{(i)} \quad \alpha = 2\pi^2\tilde{C}(E),$$

$$\begin{aligned} \text{(ii)} \quad P_\alpha(x) &= x - \frac{1}{p+2}(\Lambda\alpha)(\Lambda x) + \frac{1}{p+2}\Lambda\{\alpha(\Lambda x)\} \\ &\quad - \frac{1}{(p+1)(p+2)}\alpha(\Lambda^2 x) \\ &= x + \frac{1}{p+2}\varepsilon(\alpha, \Lambda x) + \frac{p}{(p+1)(p+2)}\alpha(\Lambda^2 x). \end{aligned}$$

Theorem (4.9) proves that if x is the harmonic dual of a complex geodesic cycle, then $P_\alpha(x)$ is primitive and orthogonal to all invariant forms. It can be readily shown that in the present situation the only invariant forms of degree (2, 2) on $\mathfrak{D}$ are spanned by κ^2 and α. Note that the image of P_α is not just the primitive forms since α is an eigenvector of P_α and not primitive. We now apply these results to (3.20). By the duality of C_η and $\hat{\omega}_\eta(p-1)$ and the linearity of P_α we denote

$$(4.15)\quad P_\alpha(C_\eta) = P_\alpha(\hat{\omega}_\eta(p-1)) = \sum_{i=1}^{l}\left[\Gamma_{\langle M_i\rangle}:\Gamma_{M_i}\right]P_\alpha\left(\hat{\omega}_{\langle M_i^{(1)}\rangle}(p-1)\right).$$

Theorem (4.10). *The family of differential forms* $\langle\phi_{\eta,\bar{s}},\Theta(\tau,h,Z)\rangle$ *has an analytic continuation to* $s=0$, *and its value at* $s=0$ *is the harmonic form*

$$(4\pi)^{-m(p+1)-2}\Gamma(p+1)^{m-1}\Gamma(p+3)N_{R/\mathbf{C}}(\eta)^{-(p+1)}\cdot P_\alpha(C_\eta),$$

where $P_\alpha(C_\eta)$ *as given by* (4.15) *is primitive and is orthogonal in the Hodge inner product to G invariant forms.*

As remarked before, the Poincaré series $\phi_{\eta,0}$ span $\mathcal{S}_{p+2}(\tilde{\Gamma})$ where $p+2=(p+2,\cdots,p+2)$. We now use Theorem (4.10) to define a lifting or correspondence:

$$(4.16)\quad \begin{aligned}&\mathbf{L}:\mathcal{S}_{p+2}(\tilde{\Gamma})\to\mathcal{H}^{2,2}(\Gamma\backslash\mathfrak{D}),\\ &\mathbf{L}(\phi)=\langle\phi,\Theta(\tau,h,Z)\rangle.\end{aligned}$$

Let $\mathbf{L}^*$ be the adjoint of $\mathbf{L}$ defined by the Petersson product on $\mathcal{S}_{p+2}(\tilde{\Gamma})$ and the Hodge inner product on $\mathcal{H}^{2,2}(\Gamma\backslash\mathfrak{D})$. For $\psi\in\mathcal{H}^{2,2}(\Gamma\backslash\mathfrak{D})$ we have by (3.15)

$$(4.17)\quad \begin{aligned}\int_{\Gamma\backslash\mathfrak{D}}\mathbf{L}\phi_{\eta,0}\wedge *\bar{\psi} &= \langle\phi_{\eta,0},\mathbf{L}^*\psi\rangle\\ &=(4\pi)^{-m(p+1)}(p!)^m N_{R/\mathbf{C}}(\eta)^{-(p+1)}a(\eta)\end{aligned}$$

where

$$\mathbf{L}^*\psi=\sum_{\substack{\eta\in S_1^*(N\mathfrak{O})\\ \eta>0}}a(\eta)e\left[\operatorname{tr}_{R/\mathbf{C}}(\eta\tau)\right].$$

It follows by (4.17) and Theorem (4.10) that

$$(4.18)\quad a(\eta)=(4\pi)^{-2}(p+1)(p+2)\int_{\Gamma\backslash\mathfrak{D}}P_\alpha(C_\eta)\wedge *\bar{\psi}.$$

Via Poincaré duality, this gives the desired geometric interpretation of the Fourier coefficients of the cusp forms in the image of $\mathbf{L}^*$ as intersection numbers.

Now define

$$(4.19)\quad \Omega(\tau, Z) = (4\pi)^{-2}(p+1)(p+2) \sum_{\substack{\eta \in S_1^*(N\mathcal{O}) \\ \eta > 0}} P_\alpha(C_\eta) e[\operatorname{tr}_{R/\mathbf{C}} \eta\tau].$$

Theorem (4.11). (i) $\mathbf{L}(\phi) = \langle \phi, \Omega(\tau, Z) \rangle$,

(ii) $\mathbf{L}^*(\psi) = \int_{\Gamma \setminus \mathfrak{D}} \Omega(\tau, Z) \wedge * \bar{\psi}$.

Proof. Let

$$K(\tau, \tau') = (4\pi)^{m(p+1)} \Gamma(p+1)^{-m} \sum_{\substack{\eta > 0 \\ \eta \in S_1^*(N\mathcal{O})}} N_{R/\mathbf{C}}(\eta)^{p+1} \phi_{\eta,0}(\tau) e\overline{\left[\operatorname{tr}_{R/\mathbf{C}}(\eta\tau')\right]}.$$

By (3.15), $K(\tau, \tau')$ is the reproducing kernel of $\mathbb{S}_{p+2}(\tilde{\Gamma})$, namely,

$$\langle K(\tau, \tau'), \phi(\tau) \rangle = \phi(\tau').$$

It follows that

$$\langle K(\tau, \tau'), \Theta(\tau, h, Z) \rangle = \Omega(\tau', Z),$$

and consequently,

$$\begin{aligned} \langle \phi(\tau'), \Omega(\tau', Z) \rangle &= \langle \phi(\tau'), \langle K(\tau, \tau'), \Theta(\tau, h, Z) \rangle \rangle \\ &= \langle \langle K(\tau', \tau), \phi(\tau') \rangle, \Theta(\tau, h, Z) \rangle = \mathbf{L}(\phi). \end{aligned}$$

This proves (i), and (ii) follows formally as in (4.17).

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